The complex Plateau problem

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Abstract. The classical Plateau problem is the research of a surface of minimal area, in the 3-dimensional Euclidean space, whose boundary is a given continuous closed curve; it has been completely solved by J. Douglas in 1929 and has been at the origin of developments of the Calculus of variations. The complex Plateau problem is analogous in a Hermitian complex manifold; it comes from a problem of holomorphic extension, from the boundary of a domain to its interior. It is a geometrical problem of extension of a closed real curve or manifold to a complex analytic variety. Since 1958, solutions are obtained under more and more general hypotheses.

Contents

1. Possible origin: holomorphic extension; polynomial envelope of a real curve.

1.1. Let X be a Hausdorff topological space whose every point has an open neighborhood homeomorphic to an open set of $\mathbb{C}^n$. Moreover, suppose that there exists a covering of X by open sets U_i, $i \in I$, and, for every $i \in I$, a homeomorphism $f_i : U_i \to$ an open set of $\mathbb{C}^n$ such that, for $U_i \cap U_j \neq \emptyset$, $f_j \circ f_i^{-1}$ be a holomorphic mapping of $f_i(U_i \cap U_j)$ onto $f_j(U_i \cap U_j)$. Then, X is called a *complex analytic manifold* of complex dimension n. A pair (U_i, f_i) is called a chart of X and U_i a coordinate open set.

1.2. In an open set of $\mathbb{C}^n$, with local complex coordinates $(z_1, \ldots, z_n)$, we define: $\overline{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \overline{z}_j} d\overline{z}_j$. We call $\overline{\partial}$-*problem*, the research of a differential form u satisfying the PDE $\overline{\partial} u = f$, where f is a differential form such that $\overline{\partial} f = 0$.

The extension theorem of Hartogs, obtained at the beginning of the 20^{th} century, has been completely proved by Bochner and Martinelli, independently in 1943. Subsequent proofs use the solution of a $\overline{\partial}$-problem. The simpler version is:

Let Ω be a bounded open set of $\mathbb{C}^n$, $n \geq 2$. Suppose that $\partial\Omega$ be of class C^k $(1 \leq k \leq \infty)$ or of class C^ω (i.e. real analytic). Let f be a function in $C^l(\partial\Omega)$, $1 \leq l \leq k$.

Then the two conditions are equivalent:

(i) *f is a CR function, i.e. the differential of f restricted to the complex subspaces of the tangent space to $\partial\Omega$, at every point, is $\mathbb{C}$-linear;*

(ii) *there exists $F \in C^l(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ such that $F \mid_{\partial\Omega} = f$.*

Then the graph of f is the boundary of the complex analytic submanifold defined by the graph of F in $\mathbb{C}^{n+1}$.

1.3. Let M be a smooth compact submanifold of dimension 1 of $\mathbb{C}^n$, we call *polynomial envelope of* M, the compact set $\{z \in \mathbb{C}^n; \mid P(z) \mid \leq \max_{\zeta \in M} \mid P(\zeta) \mid; P \in \mathbb{C}[Z]$, the polynomial ring with complex coefficients$\}$.

Then (J. Wermer (1958)), the polynomial envelope of M is either M, or the union of M with the support of a complex analytic set T, of complex dimension 1, whose boundary is M [We 58].

2. Minimality of complex analytic sets in a Kähler manifold.

2.1. *Wirtinger's inequality* (1936).

In $\mathbb{C}^n$,with complex coordinates $(z_1, \ldots, z_n)$, we have the Hermitian metric $H = \sum_{j=1}^{n} dz_j \otimes d\overline{z}_j$ and the (standard Kähler) exterior form $\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j$.

From the real vector space $I\!R^{2n} \cong \mathbb{C}^n$, we consider the real vector space $\Lambda_{2p} I\!R^{2n}$ of the $2p$-vectors with the associated norm $|.|$; every decomposable vector defines a real $2p$-plane of $\mathbb{C}^n$. We define the norm $\| \zeta \| = \inf \sum_j | \zeta_j |$ where $\zeta = \sum_j \zeta_j$, ζ_j decomposable (i.e. exterior product of elements of $I\!R^{2n}$).

Let $P_{pp} = \{ \sum_{j=1}^{N} \lambda_j \zeta_j ; \zeta_j$ decomposable defining a complex p-plane of $\mathbb{C}^n$; $\lambda_j \geq 0; N \in \mathbb{N}^* \}$.

Then, for every $\zeta \in \Lambda_{2p}\mathbb{C}^n$, we have:

$$\frac{1}{p!}\omega^p(\zeta) \leq \| \zeta \|;$$

equality uniquely for $\zeta \in P_{pp}$ [W 36].

2.2. *Kähler manifold* [K 33]. It is a complex analytic manifold X endowed with a Hermitian metric (Hermitian manifold) whose associated exterior differential 2-form ω is d-closed, i.e. $d\omega = 0$, (Kähler, 1933).

The complex projective space, with the Fubini-Study metric and the smooth projective algebraic varieties, with the induced metric are Kählerian manifolds.

2.3. We will use *currents* on X, i.e. continuous linear forms on the space of smooth differential forms with compact support, endowed with a convenient topology; on every coordinate open set, currents are differential forms with distribution coefficients.

In particuiar, we consider currents S with measure coefficients; for every compact K of X, we define the mass $W_K(S)$ of S on K. Among them, we have the space of *locally rectifiable currents* obtained as limits, in the mass topology, of the locally finite linear combinations, with integer coefficients, of C^1 images of simplices in $\mathbb{R}^p$. The integrability theorem on a complex analytic variety W of P. Lelong (1957) shows: $[W] = \int \cdot_{RegW}$ is a locally rectifiable current, d-closed, of bidimension (p, p) and positive. The integration current $[V]$ on a complex analytic variety is such that $M_K[V]$ is the volume de $V \cap K$.

From Wirtinger'inequality, for every d-closed compactly supported current with measure coefficients R

$$(*) \qquad M_K[V] \leq M(\chi_K[V] + R)$$

where χ_K is the characteristic function of K.

3. Boundary problem as a problem of geometric extension: Plateau problem in Kählerian case.

3.1. *Holomorphic chains*. A holomorphic chain of an open set Ω of a complex manifold X is a locally rectifiable, d-closed current of bidimension (p,p), $T = \sum n_j[W_j]$, where W_j is an irreducible complex analytic variety, of complex dimension p, $n_j \in \mathbb{Z}$, the sum being locally finite.

3.2. *Boundary problem of holomorphic chains.* On a Hermitian manifold X of complex dimension n, let M be a smooth, oriented, closed real $(2p-1)$-dimensional submanifold , $0 < p \leq n$. We still denote M the integration current $[M] = \int \cdot_M$ on M; then M is of locally finite mass and $dM = 0$. If there exists a holomorphic p-chain T of $X \setminus M$, of locally finite mass in the neighborhod of M, M has a simple (or trivial) extension to X, still denoted T and if $dT = M$, we say that M is the boundary of T.

M being given on X, we look for necessary and sufficient conditions for M to be the boundary of a holomorphic p-chain: this is the *boundary problem for* M *in* X.

In the situation of section 1.2 (graph of the functions in the Hartogs' theorm), the solution of the boundary problem appears as an extension of gr f into a complex analytic variety translating the extension of f to F. The same conclusion is true in the general case if all the coefficients of the holomorphic chain are equal to 1.

3.3. *Strict Plateau problem in Kählerian case.*

When all coefficients of T are equal to 1, from the consequence (*) (section 2.3) of the Wirtinger inequality, on a Kählerian manifold, the volume of the analytic variety supp T (identified with T) is minimal; it is a relative minimum because, in general, the solution of the boundary problem is not unique. We have obtained a solution of the complex Plateau problem.

In the following, we shall be interested in the solution of the boundary problem for M in X, a generalization of the Plateau problem.

4. Solutions of the boundary problem in different spaces.

4.1. The first result has been obtained in 1958, by J. Wermer, in $\mathbb{C}^n$, for $p = 1$ and M holomorphic image of the unit circle in $\mathbb{C}$ [We 58]; this result has been generalized to the case where M is a union of C^1 real connected curves by Bishop, Stolzenberg (1966), looking for the polynomial envelope of M according to section 1.3.

In $\mathbb{C}^n$, after preliminary results by Rothstein (1959) [Rs 59], the boundary problem has been solved by Harvey and Lawson (1975), for $p \geq 2$, under the necessary and sufficient condition: M is compact, maximally complex and, for $p = 1$, under the moment condition: $\int_M \varphi = 0$, for every holomorphic 1-form φ on $\mathbb{C}^n$ [HL 75]. For $n = p+1$, the method, inspired by the Hartogs' theorem consists in building T as the divisor of a meromorphic function; this function itself is constructed, step by step, from solutions of $\overline{\partial}$-problems with compact support. T can also be viewed as graph (with multiplicities on the irreducible components) of an analytic function with a finite number of determinations. For any p, we come back to the particular case using projections.

In $\mathbb{C}\mathsf{P}^n \setminus \mathbb{C}\mathsf{P}^{n-r}$, $1 \leq r \leq n$, for compact M, the problem has a solution if and only if, for $p \geq r+1$, M is maximally complex and if, for $p = r$, M satisfies the moment condition: $\int_M \varphi = 0$, for every $\overline{\partial}$-closed $(p, p-1)$-forme φ . The method consists in solving the boundary problem, in $\mathbb{C}^{n+1} \setminus \mathbb{C}^{n-r+1}$, for the inverse image of M by the canonical projection [HL 77].

In both cases, the solution is unique.

Harvey et Lawson assume the given M to be, except for a closed set of Hausdorff $(2p-1)$-dimensional measure zero, an oriented manifold of class C^1; we shall say: M is a variety C^1 with negligible singularities.

E. Chirka, in 1985, for $p \geq 2$ and M relatively compact, solved the boundary problem in the complement of a holomorphically convex compact in $\mathbb{C}^n$ [Ch 89].

The boundary problem in $\mathbb{C}\mathrm{P}^n$ has been set up, for the first time, by J. King [Ki 79]; uniqueness of the solution is no more possible, since two solutions differ by an algebraic p-chain.

4.2. In $\mathbb{C}\mathrm{P}^n$, a solution of the boundary problem has been obtained by P. Dolbeault et G. Henkin for $p = 1$, (1994), then for every p, (1997) and more generally, in a q-linearly concave domain X of $\mathbb{C}\mathrm{P}^n$, i.e. a union of projective subspaces of dimension q [DH 97].

The necessary and sufficient condition for the existence of a solution is an extension of the moment condition: it uses a Cauchy residue formula in one variable and a non linear differential condition which appears in many questions of Geometry or mathematical Physics. In the simplest case: $p = 1$, $n = 2$, this is the shock wave equation for a local holomorphic function in 2 variables ξ, η, $f\dfrac{\partial f}{\partial \xi} = \dfrac{\partial f}{\partial \eta}$.

From a local condition, the above relation allows to construct, by extension ot the coefficients, a meromorphic function playing, in $\mathbb{C}^n$, the same part as the Harvey-Lawson function described above; it defines a holomorphic p-chain extendable to $\mathbb{C}\mathrm{P}^n$ using the classical Bishop-Stoll theorem.

For technical reasons, and for lack of using slicing, the regularity is supposed to be C^2.

4.2.1. Moreover, as a consequence, and thanks to a compacity theorem of Sacks-Uhlenbeck, we have: if the sections of M by almost all projective $(p+1)$-codimensional subspaces, belonging to a small enough open set of the Grassmanniann, are the boundaries of holomorphic 1-chains, then M is the boundary of a holomorphic p-chain of $\mathbb{C}\mathrm{P}^n$ or of X.

The conditions of regularity of M have been weakened, first in $\mathbb{C}^n$, and for $p = 1$, to a condition, a little stronger than the rectifiability, par H. Alexander [Al 88] who, moreover, has given an essential counter-example [Al 87], then by Lawrence [Lce 95] and finally, and for any p, in $\mathbb{C}^n$ and $\mathbb{C}\mathrm{P}^n$, by T.C. Dinh [Di 98]: M is a rectifiable current whose tangent cone is a vector subspace almost everywhere. Moreover, Dinh has obtained the reduction of the boundary problem in $\mathbb{C}\mathrm{P}^n$ to the case $p = 1$, with weaker conditions than above and by an elementary analytic procedure [Di 98].

All the previous results are obtained as Corollaries.

5. A topological condition in $\mathbb{C}^n$.

H. Alexander et J. Wermer [A-We 2000] got a solution of the boundary problem in $\mathbb{C}^n$ without explicit analytic condition.

5.1. *Linking number.*

Let M be a compact, oriented submanifold of odd dimension k, of $\mathbb{C}^n$ and A be an algebraic subvariety of $\mathbb{C}^n$ of complex dimension s such that $k+2s=2n-1$, endowed with its natural orientation. A being non compact, we replace it by a compact variety A' coinciding with A in the interior of a large enough ball. Let Σ an oriented, compact $(s+1)$-subvariety of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ such that $M = b\Sigma$, the linking coefficient of M and A is

$$\text{link}(M, A) = \sharp(\Sigma, A'),$$

intersection number of Σ and A'.

5.2. *Theorem.- Let M be as above such that $3 \leq k \leq 2n-3$. Then the following conditions are equivalent:*

(i)
$$\text{link}(M, A) \geq 0$$

for every algebraic subvariety of $\mathbb{C}^n$ disjoint from M, of pure complex dimension $n - \frac{k+1}{2}$;

(ii) *M is maximally complex and* (from Harvey-Lawson) *there exists a unique holomorphic $\frac{k+1}{2}$-chain , in $\mathbb{C}^n \setminus M$, of finite mass, with bounded support, such that $M = bT$.*

Moreover, for every $x \in \mathbb{C}^n \setminus M$, $x \in \text{supp } T$ is equivalent to
$$\text{link}(M, A) > 0$$

for every A as above, such that $x \in A$.

5.3. *Proof.* Using Harvey-Lawson, first for $k = 1$ and the moment condition, then for $k = 3$, finally for the general case, by slicing. An essential lemma, for $k = 1$, $M = \gamma$ is:

$$\mathrm{link}(\gamma, A) = \frac{1}{2\pi i} \int_\gamma \frac{dP}{P}$$

where $P \in \mathbb{C}[Z]$ and $A = Z(P)$.

6. Boundary problem of Levi-flat hypersurfaces in $\mathbb{C}^n$.

6.1. A *Levi-flat hypersurface in* $\mathbb{C}^n$ is foliated by complex hypersurfaces of $\mathbb{C}^n$ or equivalently, possesses an everywhere vanishing Levi form. It is an odd dimensional real manifold (or variety) very close to a complex manifold (or variety).

Let $S \subset \mathbb{C}^n$ be a compact connected 2-codimensional submanifold. Find a Levi-flat hypersurface $M \subset \mathbb{C}^n \setminus S$ such that $dM = S$ (i.e. whose boundary is S, possibly as current).

For $n = 2$, near an elliptic complex point $p \in S$, $S \setminus \{p\}$ is foliated by smooth compact real curves which bound analytic discs (Bishop; '65). The family of these discs fills a smooth Levi-flat hypersurface.

In 1983, Bedford-Gaveau [BeG 83] considered the case of a particular sphere with two elliptic complex points. If S is contained in the boundary of a strictly pseudoconvex bounded domain, then the families of analytic discs in the neighborhood of each elliptic point extend to a global family filling a 3-dimensional ball M bounded by S.

In 1991, Bedford-Klingenberg [BeK 91] and Kruzhilin [K 91], with independent methods, extended the result when there exist hyperbolic complex points on S with the same global condition.

Also, by the methods of complex geometric theory, results of increasing generality have been obtained by [Sh 93], [ChS 95], [SIT 94] and finally in the non compact case by [ShT 99]. The global sufficient condition of embedding of S in the boundary of a strictly pseudoconvex domain is still required in these papers.

A first result for $n \geq 3$ (in the sense of currents), and for elliptic points only, has recently been obtained [DTZ 05]; a new result, partially conjectural, in the particular case where S is homeomorphic to a sphere, for three elliptic and one hyperbolic points is described here. A local condition is required because, in general, S is not locally the boundary of a Levi-flat hypersurface. The proof uses Thurston's stability theorem for foliations on S, and a parametric version of the Harvey-Lawson theorem on boundaries of complex analytic varieties. There is no global condition.

6.2. Let M be a C^∞ smooth, connected, real submanifold of $\mathbb{C}^n$, $n \geq 2$, of real dimension m. For every $p \in M$ we denote $H_p(M) = T_p(M) \cap iT_p(M)$ the complex tangent space of M at p and HM the complex tangent fiber bundle; $\dim_{CR} M_p := \dim_{\mathbb{C}} H_pM$ is called the CR *dimension* of M at p. M is said to be a CR *manifold* if $\dim_{\mathbb{C}} H_pM$ is constant. In this case $\dim_{CR} M := \dim_{\mathbb{C}} H_pM$ is, by definition, the CR *dimension* of M. The CR dimension satisfies $m - n \leq \dim_{CR} M \leq m/2$. A CR submanifold $M \subset \mathbb{C}^n$ of CR dimension $m - n$ is called *generic*.

A CR submanifold $M \subset \mathbb{C}^n$ is called *minimal* at a point p if there does not exist a submanifold N of M of lower dimension through p such that $HN = HM|_N$. By a theorem of Sussman, all possible submanifolds N such that $HN = HM|_N$ contain, as germs at p, one of the minimal possible dimension, called a CR *orbit* of p in M. The germ at p of the CR orbit of p is uniquely determined. In particular, M is minimal at p if and only if a CR orbit of p is open in M and hence, of the maximal possible dimension. The minimal possible dimension of a CR orbit is $\dim_{\mathbb{R}} H_p M = 2$ CR $\dim M$.

6.3. *Behavior at* CR *points.*[DTZ 05]

Let S be a smooth real submanifold of real codimension 2 in $\mathbb{C}^n$. We say that S is a *locally flat boundary* at a point $p \in S$ if an open neighborhood of p in S locally bounds a Levi-flat hypersurface $M \subset \mathbb{C}^n$.

At CR-points, this condition is equivalent to: S *is non minimal*.

6.4. *Behaviour at complex points.* Near a complex point $p \in S$, i.e. p is such that T_pS is a complex hyperplane in $T_p\mathbb{C}^n$. In suitable holomorphic coordinates $(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at p, S is locally given by an equation

(1)
$$w = Q(z) + O(|z|^3), \quad Q(z) = \sum_{1 \le i,j \le n-1} (a_{ij} z_i z_j + b_{ij} z_i \overline{z}_j + c_{ij} \overline{z}_i \overline{z}_j)$$

where (a_{ij}) and (c_{ij}) are symmetric complex matrices and (b_{ij}) is an arbitrary complex matrix.

We call S *flat* at a complex point $p \in S$ if, in some (and hence in any) coordinates (z,w) as in (1), the term $\sum b_{ij} z_i \overline{z}_j$ takes values in some real line in $\mathbb{C}$, i.e. there exists a complex number $\lambda \in \mathbb{C}$ such that $\sum b_{ij} z_i \overline{z}_j \in \lambda \mathbb{R}$ for all $z = (z_1, \ldots, z_{n-1})$.

Let $S \subset \mathbb{C}^n$ be a locally flat boundary with complex point $p \in S$. Then S is flat at p.

We say (and we assume in the following) that S is in a *flat normal form* at p if the coordinates (z, w) as in (1) are chosen such that $Q(z) \in \mathbb{R}$ for all $z \in \mathbb{C}^{n-1}$.

6.5. *Elliptic or hyperbolic flat points.*

We assume the above necessary conditions for S in order to be a locally flat boundary: S is nonminimal in its generic points and S is flat at complex points.

The quadratic form:

$$(1') \qquad Q(z) = z\overline{z} + \sum_{j=1}^{n-1} \lambda_j \mathcal{R}e\ z_j^2, \quad \lambda_j \geq 0$$

is a particular case of the quadratic form $Q(z)$ in (1) for S in a flat normal form; in particular, for $n = 2$, the two foms in (1) and (1') coincide up to a linear change of coordinates.

Let $p \in S$ be a flat point. We say that p is *elliptic* if the real quadratic form $Q(z)$ is positive or negative definite in (1); in the particular case of (1'), $\quad 0 \leq \lambda_j < 1$.

This generalizes the well-known notion of ellipticity (in the sense of Bishop) for $n = 2$.

Let $p \in S$ be a flat point and $J \subset \{1, \ldots, n-1\}$. We say that p is *J-hyperbolic* if the real quadratic form $Q(z)$ of formula (1') is such that $1 < \lambda_j$ for $j \in J$ and $0 \leq \lambda_j < 1$ for $j \in \{1, \ldots, n-1\} \setminus J$.

This is a generalization of a hyperbolic point in [BeK 91].

We will not consider *parabolic points* (one $\lambda_j = 1$ at least) which don't appear generically.

6.5.1. *Proposition* [DTZ 05]. *Assume that $S \subset \mathbb{C}^n$, ($n \geq 3$) is nowhere minimal at all its CR points and has an elliptic flat complex point p. Then there exists a neighborhood V of p such that $V \setminus \{p\}$ is foliated by compact real* ($2n-3$)*-dimensional CR orbits diffeomorphic to the sphere $\mathbf{S}^{2n-3}$ and there exists a Lipschitz function ν, smooth and without critical points away from p, having the CR orbits as the level surfaces.*

In the following, we shall call *hyperbolic* any J-hyperbolic point with $J \neq \emptyset$.

Lemma. *Suppose that the quadric* (2) *is flat and that* 0 *is a hyperbolic point. Then it is CR and nowhere minimal outside* 0, *and the CR orbits are the* $(2n-3)$*-dimensional submanifolds given by* $w = const$.

The section $w = 0$ is tangent to a real cone in $\mathbb{R}^{2n}$ whose vertex is 0.

6.5.3. *Corollary*. *For general* S *with* $p \in S$ *an isolated hyperbolic point, we find two disjoint orbits* σ_1, σ_2 *in* $S \setminus \{p\}$ *whose union with the point* p *is connected.*

6.5.4. *Induced foliation by CR orbits: case of elliptic flat points.*

Let $S \subset \mathbb{C}^n$ *be a compact connected real 2-codimensional manifold such that the following holds:*

(i) S *is nonminimal at every CR point;*

(ii) *every complex point of* S *is flat and elliptic and there exists at least one such point;*

(iii) S *does not contain complex manifold of dimension* $(n-2)$*.*

Then S is homeomorphic to the unit sphere $S^{2n-2} \subset \mathbb{C}_z^{n-1} \times \mathbb{R}_x$ such that the complex points are the poles $\{x = \pm 1\}$ and the CR orbits in S correspond to the $(2n-3)$-spheres given by $x =$ const. In particular, if S_{ell} denotes the (finite) set of all elliptic flat complex points of S, the open subset $S_0 = S \setminus S_{\text{ell}}$ carries a foliation $\mathfrak{F}$ of class C^∞ with 1-codimensional CR *orbits as compact leaves.*

6.5.5 *Induced foliation by CR orbits: case of hyperbolic flat points.*

Let $S \subset \mathbb{C}^n$ *be a compact connected real 2-codimensional manifold such that the following holds:*

(i) S *is nonminimal at every CR point;*

(ii) S *is homeomorphic to a sphere and contains, as complex points, exactly three elliptic flat points and one hyperbolic flat point; the closures of the separatrix orbits* $\overline{\sigma}_1$ *and* $\overline{\sigma}_2$ *are compact;*

(iii) S *does not contain complex manifold of dimension* $(n-2)$.

Then, in the notations of Corollary 6.5.3, $S \setminus \overline{\sigma}_1 \cup \overline{\sigma}_2$ *is divided into three disjoint connected domains* S_1', S_2', S_3' *whose closures* S_1, S_2, S_3 *are oriented manifolds with boundaries* $\overline{\sigma}_1^+, \overline{\sigma}_2^+$ *for the first two and with corners for the third one, the boundary of* S_3 *beeing* $\overline{\sigma}_1^- \cup \overline{\sigma}_2^-$. *Moreover,* S_j *contains a unique elliptic point* e_j *in its interior. Each* $S_j' \setminus e_j$, $j = 1, 2, 3$ *carries a foliation* $\mathfrak{F}_j$ *of class* C^∞ *with* 1*-codimensional* CR *orbits as compact leaves.*

The proofs of Propositions 6.5.4 and 6.5.5 use the

Thurston's Stability Theorem: ([CaC], Theorem 6.2.1). *Let* $(M, \mathfrak{F})$ *be a compact, connected, transversely-orientable, foliated manifold with boundary or corners, of codimension 1, of class* C^1*. If there is a compact leaf* L *with* $H^1(L, \mathbb{R}) = 0$*, then every leaf is homeomorphic to* L *and* M *is homeomorphic to* $L \times [0, 1]$*, foliated as a product.*

Remark.- The hypothesis (ii) of the Proposition 6.5.5 suppose that the situation for $n = 2$ is still valid for $n \geq 3$: up to now, this is a *conjecture*. It should be necessary to evaluate the difference between the number of elliptic points and the number of hyperbolic points when S has dimension at least 4 from the topology of the manifold S. In the same way, the existence of compact $\overline{\sigma}_1$ and $\overline{\sigma}_2$ is conjectural.

6.6. *Boundary value problem for embedded surfaces.*

When does a compact submanifold S of $\mathbb{C}^n$ bounds a Levi-flat hypersurface M? From Propositions 6.5.4 and 6.5.5, every CR orbit of S is a connected compact maximally complex CR submanifold of $\mathbb{C}^n$, $n \geq 3$, and hence, in view of Harvey-Lawson [H 77], bounds a complex-analytic subvariety. Thus, in order to find M, at least as a real "subvariety", foliated by complex subvarieties, we build it as a family of the solutions of the boundary problems for individual CR orbits. To do it, we reduce the problem to the corresponding problem in a real hyperplane of $\mathbb{C}^{n+1}$. The latter case is treated in the next section.

6.7. *On boundaries of families of holomorphic chains with a C^∞ parameter.* [DTZ 05]

As in [Do 86] we follow the method of Harvey-Lawson in [H 77, Section 3].

6.7.1. *Boundary problem in a real hyperplane of $\mathbb{C}^n$.*

Let $n \geq 4$. We shall use the following notations: $z'' = (z_2, \ldots, z_{n-1}) \in \mathbb{C}^{n-2}$, $\zeta' = (x_1, z'') \in I\!R \times \mathbb{C}$. Let $E = I\!R \times \mathbb{C}^{n-1} = \{y_1 = 0\} \subset \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$, and $k : E \to I\!R_{x_1}$, $(x_1; z''; z_n) \mapsto x_1$. For $x_1^0 \in I\!R_{x_1}$, let $E_{x_1^0} = k^{-1}(x_1^0)$.

Let $N \subset E$ be a compact, (oriented) CR subvariety of $\mathbb{C}^n$ of real dimension $2n-4$ and CR dimension $n-3$, $(n \geq 4)$, of class C^∞, with negligible singularities (i.e. there exists a closed subset $\tau \subset N$ of $(2n-4)$-dimensional Hausdorff measure 0 such that $N \setminus \tau$ is a CR submanifold in $E \setminus \tau$). Assume that N, as a current of integration, is d-closed and satisfies:

(H) there exists a closed subset $\tau' \supset \tau$ of N with $\mathcal{H}^{2n-4}(\tau') = 0$ such that for every $z \in N \setminus \tau'$, $N \setminus \tau'$ is a submanifold transversal to the maximal complex affine subspace of E through z;

(H′) there exists a closed subset $L_0 \subset \mathbb{R}_{x_1}$ with $\mathcal{H}^1(L_0) = 0$ such that for every $x_0 \in k(N) \setminus L_0$, the fiber $k^{-1}(x_0) \cap N$ is connected.

For every $x_1 \in k(N)$, let $N_{x_1} = N \cap E_{x_1}$ and consider the points $z \in N_{x_1}$ satisfying the following conditions

(i) either $z \in \tau$; or

(ii) E_{x_1} is not transverse to N at z.

Let τ'_{x_1} be the set of such points in N_{x_1} and

$$L := L_0 \cup \{x_1 \in \mathbb{R} : \mathcal{H}^{2n-5}(\tau'_{x_1}) > 0\}.$$

Lemma. $\mathcal{H}^1(L) = 0$.

6.7.2. *Theorem. Let* $n \geq 4$*, let* N *satisfy* (H) *and* (H′)*, and* L *be chosen, as above. Then, there exists, for* $E' = E \setminus k^{-1}(L)$*, a unique* C^∞ *maximally complex* $(2n-3)$*-subvariety* M *with negligible singularities in* $E' \setminus N$*, foliated by complex* $(n-2)$*-subvarieties, with the properties that* M *simply (or trivially) extends to* E' *by a* $(2n-3)$*-current (still denoted* M*) such that* $dM = N$ *in* E'*. The leaves are the sections by the hyperplanes* $E_{x_1^0}$*,* $x_1^0 \in k(N) \setminus L$*, and are the solutions of the "Harvey-Lawson problem" for finding a holomorphic subvariety in* $E_{x_1^0} \cong \mathbb{C}^{n-1}$ *with prescribed boundary* $N \cap E_{x_1^0}$*.*

6.8. *On some Levi-flat* $(2n-1)$*-subvarieties with given boundary in* $\mathbb{C}^n$*.*

Using a function ν whose level sets are the leaves of the foliation $\mathcal{F}$, we translate this problem into a boundary problem for subvarieties of a hyperplane E of $\mathbb{C}^{n+1}$ with negligible singularities, and then apply Theorem 6.7.2.

6.8.1. *Theorem [DTZ 05]. Let* $S \subset \mathbb{C}^n$, $n \geq 3$, *be a compact connected smooth real 2-codimensional submanifold such that the following holds:*

(i) S *is nonminimal at every CR point;*

(ii) *every complex point of* S *is flat and elliptic and there exists at least one such point;*

(iii) S *does not contain complex submanifolds of dimension* $(n-2)$.

Then there exists a Levi-flat $(2n-1)$*-subvariety* $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ *with boundary* $\tilde{S}$ *(in the sense of currents) such that the natural projection* $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ *restricts to a bijection which is a CR diffeomorphism between* $\tilde{S}$ *and* S *outside the complex points of* S.

6.8.2. *Theorem. Let* $S \subset \mathbb{C}^n$, $n \geq 3$, *be a compact connected smooth real* 2*-codimensional submanifold such that the following holds:*

(i) S *is a topological sphere;* S *is nonminimal at every CR point;*

(ii) *every complex point of* S *is flat ; there exist three elliptic points* $e_j, j = 1, 2, 3$ *and one hyperbolic point* h*;*

(iii) S *does not contain complex submanifolds of dimension* $(n-2)$*.*

Then there exists a Levi-flat $(2n-1)$*-subvariety* $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ *with boundary* $\tilde{S}$ *(in the sense of currents) such that the natural projection* $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ *restricts to a bijection which is a CR diffeomorphism between* $\tilde{S}$ *and* S *outside the complex points of* S*.*

References

[Al 88] H. Alexander, The polynomial hull of a rectifiable curve in $\mathbb{C}^n$, Amer. J. Math., **110** (1988), 629-640.

[A-We 2000] H. Alexander and J. Wermer, Linking numbers and boundaries of varieties. Ann. of Math. (2) **151** (2000), 125-150.

[BeG 83] E. Bedford & B. Gaveau, *Envelopes of holomorphy of certain 2-spheres in* $\mathbb{C}^2$, Amer. J. Math. **105** (1983), 975-1009.

[BeK 91] E. Bedford & W. Klingenberg, *On the envelopes of holomorphy of a 2-sphere in* $\mathbb{C}^2$, J. Amer. Math. Soc. **4** (1991), 623-646.

[Bi 65] E. Bishop, Differentiable manifolds in complex Euclidean space, Duke Math. J. **32** (1965),1-22.

[Ch 89], E.M. Chirka, *Complex analytic sets*, Mathematics and its applications, **46**, Kluwer Academic Publishers, 1989; Rusian edition 1985.

[ChS 95] E. M. Chirka & N. V. Shcherbina, *Pseudoconvexity of rigid domains and foliations of hulls of graphs.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22** (1995), no. 4, 707–735.

[Di 98] T.C. Dinh, Enveloppe polynomiale d'un compact de longueur finie et chaînes holomorphes à bord rectifiable, Acta Math. **180** (1998), 31-67.

[Do 86] P. Dolbeault, *Sur les chaînes maximalement complexes de bord donné*, Proc. Symp. Pure Math. **44**, Amer. Math. Soc. (1986), 171-205.

[DH 97] P. Dolbeault et G. Henkin, Chaînes holomorphes de bord donné dans $\mathbb{C}P^n$, Bull. Soc. Math. France **125** (1997), 383-445.

[DTZ 05] P. Dolbeault, G. Tomassini, D. Zaitsev, *On boundaries of Levi-flat hypersurfaces in* $\mathbb{C}^n$, C.R. Acad. Sci. Paris, Ser. I **341** (2005) 343-348

[D 31] J. Douglas, Solution of the problem of Plateau, Trans. Am. Math. Soc. **33** (1931), 263-321.

[H 77] R. Harvey, *Holomorphic chains and their boundaries*, Proc. Symp. Pure Math. **30**, Part I, Amer. Math. Soc. (1977),309-382.

[HL 77] R. Harvey and B. Lawson, On boundaries of complex analytic varieties, II, Ann. of Math., **106**, (1977), 213-238.

[Ki 79] J. King, Open problems in geometric function theory, Proceedings of the fifth international symposium of Math.,p. 4, The Taniguchi foundation, 1978.

[K 91] N. G. Kruzhilin, *Two-dimensional sphere in the boundary of strictly pseudoconvex domains in* $\mathbb{C}^2$, Izw. Akad Nauk SSSR, Ser. Math., Tom 55 (1991), n. 6 and Math. USSR-Izv. **39** (1992), n. 3, 1151-1187.

[Lce 95] M.G. Lawrence, Polynomial hulls of rectifiable curves, Amer. J. Math., **117** (1995), 405-417.

[Le 57] P. Lelong, Intégration sur un ensemble analytique complexe, Bull. Soc. Math. France, **85** (1957),

[Rs 59] W. Rothstein, Bemerkungen zur Theorie komplexer Räume, Math. Ann., **137** (1959), 304-315.

[Sh 93] N. Shcherbina, *On the polynomial hull of a graph*, Indiana Univ. Math J. **42** (1993), 477-503.

[ShT 99] N. Shcherbina & G. Tomassini, *The Dirichlet problem for Levi flat graphs over unbounded domains*, Internat. Math. Res. Notices (1999), 111-151.

[SlT 94] Z. Slodkowski & G. Tomassini, *The Levi equation in higher dimension and relationship to the envelope of Holomorphy*, Amer. J. Math. **116** (1994), n. 2, 479-499.

[We 58] J. Wermer, The hull of a curve in $\mathbb{C}^n$. Ann.of Math. **68** (1958), 550-561.

[W 36] W. Wirtinger, Ein Determinantenidentität und ihre Anwendung auf analytische Gebilde und Hermitische Massbestimmung. Monatsh. f. Math. u. Physik, **44** (1936), 343-365.