Introduction of the $\overline{\partial}$-cohomology

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Abstract. We recall results, by Hodge during the thirties, the early forties and 1951, by A.Weil (1947 and 1952), on differential forms on complex projective algebraic and Kähler manifolds; then we describe the appearance of the $\overline{\partial}$-cohomology in relation to the cohomology of holomorphic forms.

Contents

1.1. In [H 41] and former papers, Hodge defined harmonic differential forms on a Riemannian manifold X; using the Riemannian metric, he defined, on differential forms, the dual δ of the exterior differential operator d, the Laplacian $\Delta = d\delta + \delta d$, harmonic forms ψ satisfying $\Delta\psi = 0$ and proved the following *decomposition theorem: every differential form* $\varphi = \mathcal{H}(\varphi) + d\alpha + \delta\beta$ and, from de Rham's theorem: $H^p(X, \mathbb{C}) \cong \mathcal{H}^p(X)$.

[H 41] W.V.D. Hodge, The theory and applications of harmonic integrals, (1941), 2th edition 1950.

Then Hodge gave applications to smooth complex projective algebraic varieties (chapter 4), the ambient projective space being endowed with the Fubini-Study hermitian metric: Hodge theory mimics the results of Lefschetz [L 24], via the duality between differential forms and singular chains. The complex local coordinates being $(z_1, \ldots, z_n)$, Hodge uses the coordinates $(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n)$ for the C^∞, or C^ω functions and the *type* (with a slight different definition) (p, q) for the differential forms homogeneous of degree p in the dz_j, and q in the $d\overline{z}_j$.

1.2. In a letter to G. de Rham in 1946 [W 47], A. Weil states that the results of ([H 41], chapter 4) are true for a compact Kähler manifold and studies the following situation for closed meromorphic differential forms of degree 1 on a compact Kähler manifold V:

Let $r = (U_j)$ be a locally finite covering of V by open sets U_j such that U_j and $U_j \cap U_k \neq \emptyset$ be homeomorphic to open balls. For every j, let θ_j be a d-closed meromorphic 1-form on U_j such that on every $U_j \cap U_k \neq \emptyset$, $\theta_j - \theta_k = \theta_{jk}$ is holomorphic. Remark that:

$$\theta_{lj} + \theta_{jk} + \theta_{kl} = 0 \text{ and } d\theta_{jk} = 0$$

[W 47] A. Weil, Sur la théorie des formes différentielles attachées à une variété analytique complexe, Comment. Math. Helv., **20** (1947), 110-116.

The problem is to find a closed meromorphic 1-form θ having the singular part θ_j on U_j for any j. Using a result of Whitney, we construct smooth 1-forms ζ_j on U_j such that $\zeta_j - \zeta_k = \theta_{jk}$ in the following way: assume already defined the forms $\zeta_1, \ldots, \zeta_{k-1}$, ζ_k is a C^∞ extension of $\zeta_{k-1} - \theta_{(k-1)k}$ from $U_{k-1} \cap U_k$ to U_k. Then, there exists, on V a smooth 1-form $\sigma = d\zeta_j$ on U_j; using the existence theorem of harmonic forms, we show that σ is harmonic of type $(1, 1)$. The existence of θ is equivalent to $\sigma = 0$.

Moreover remark that the 1-cocycle $\{\theta_{jk}\}$ defines a fibre bundle [Ca 50].

1.3. More generally, let $\{u_{jk}\}$, where u_{jk}, $p \geq 0$, is a d-closed holomorphic p-form, be a 1-cocycle of the nerve of the covering r, then u_{jk} is a holomorphic p-form on $U_j \cap U_k$ and $u_{lj} + u_{jk} + u_{kl} = 0$ on $U_j \cap U_k \cap U_l \neq \emptyset$. As above, there exist C^∞ $(p, 0)$-forms g_j on U_j such that $g_j - g_k = u_{jk}$ and a harmonic form $\mathcal{L}^{p,1}$ on V such that $dg_j = \mathcal{L}^{p,1}_{|U_j}$. Conversely, on U_j (small enough), from the Poincaré lemma, there exists, a $(p, 0)$-form g_j such that $dg_j = \mathcal{L}^{p,1}_{|U_j}$,then $\{u_{jk}\} = \{g_j - g_k\}$ is a 1-cocycle with u_{jk} holomorphic [Do 51].

1.4. In [H 51], Hodge defined the differential operator $d'' = \sum_{j=1}^{n} \frac{\partial}{\partial \overline{z}_j} d\overline{z}_j$ of type $(0,1)$; let $d' = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j$ of type $(1,0)$; then $d = d' + d''$ and $d''^{\,2} = 0 = d'^2$.

After [Ca 51], the use of d', d'' and, on Kähler manifolds, the operators δ' and δ'' became usual.

2. First unpublished proof of the isomorphism.

2.1. Let X be a paracompact (in particular countable union of compact sets) complex analytic manifold of complex dimension n. Let $r = (U_j)$ be a locally finite covering of X by open sets U_j such that U_j and $U_j \cap U_k \neq \emptyset$, or more generally $\bigcap_{j \in J} U_j \neq \emptyset$ for $J \subset (1, 2, \ldots, n)$ be homeomorphic to open balls. It is always possible to replace r by a covering $r' = (U'_j)$ s.t. $\overline{U}'_j \subset U_j$. We will use $\check{C}$ech cochains, cocycles and cohomology.

As in section 1.3, let $\{u_{jk}\}$ be a 1-cocycle of the nerve N_r of r where u_{jk} is a holomorphic p-form on U_{jk}, with $p \geq 0$ and $u_{lj} + u_{jk} + u_{kl} = 0$ on $U_{jkl} = U_j \cap U_k \cap U_l \neq \emptyset$.. Then the $(p,0)$-forms u_{jk} satisfy $d'' u_{jk} = 0$. As above, there exist g_j C^∞ $(p,0)$-forms such that $g_j - g_k = u_{jk}$: then there exists a global d''-closed $(p,1)$-form h such that $h_{|U_j} = d'' g_j$.

Conversely, given h on X, such that $d'' h = 0$, then, on U_j (small enough), from the d''-lemma (see section 3), there exists, a $(p,0)$-form g_j such that $d'' g_j = h_{|U_j}$,then $\{u_{jk}\} = \{g_j - g_k\}$ is a 1-cocycle with u_{jk} holomorphic.

2.2. Let $\mathcal{E}^{p,q}$ be the sheaf of differential forms (or currents) of type (p,q) a complex analytic manifold X.

$$\begin{aligned} Z^{p,q}(X,\mathbb{C}) &= \mathrm{Ker}(\mathcal{E}^{p,q}(X) \xrightarrow{d''} \mathcal{E}^{p,q+1}(X)) \\ B^{p,q}(X,\mathbb{C}) &= \mathrm{Im}(\mathcal{E}^{p,q-1}(X) \xrightarrow{d''} \mathcal{E}^{p,q}(X)) \end{aligned}$$

We call d''-cohomologie group of type (p,q) of X, the $\mathbb{C}$-vector space quotient

$$H^{p,q}(X,\mathbb{C}) = Z^{p,q}(X,\mathbb{C})/B^{p,q}(X,\mathbb{C})$$

2.3. Let Ω^p the sheaf of holomorphic differential p-forms. From section 2.2, we have the isomorphism: $H^1(\Omega^p) \cong H^{p,1}(X,\mathbb{C})$.

2.4. Let now $\{u_{jkl}\}$ be a 2-cocycle of the nerve N_r of the covering r, where u_{jkl} is a holomorphic p-form, we have: $u_{mjk} + u_{jkl} + u_{klm} + u_{lmj} = 0$ on $U_{jklm} = U_j \cap U_k \cap U_l \cap U_m \neq \emptyset$. The $(p,0)$-forms u_{jkl} satisfy $d'' u_{jkl} = 0$. As above, there exist g_{jk} C^∞ $(p,0)$-forms such that $g_{lj} + g_{jk} + g_{kl} = u_{jkl}$ on $U_{jkl} \neq \emptyset$, then $d'' g_{lj} + d'' g_{jk} + d'' g_{kl} = 0$ on $U_{jkl} \neq \emptyset$, and three other analogous equations, the four homogenous equations are valid on U_{jklm}. Then: $d'' g_{lj} = 0$; $d'' g_{jk} = 0$; $d'' g_{kl} = 0$; $d'' g_{lm} = 0$ on U_{jklm}. If U_{jklm} is small enough, from the d''-lemma (see section 3), there exists, h_{jk} such that $d'' h_{jk} = g_{jk}$ on U_{jklm}. The form h_{jk} can be extended to U_{jk} such that $h_{jk} + h_{kl} + h_{lj} = 0$ on U_{jkl}; by convenient extension, there exists a form μ_j on U_j such that $\mu_j - \mu_k = d'' h_{jk}$ on U_{jk}, and a d''-closed $(p,2)$-form λ on X such that $d'' \mu_j = \lambda_{|U_j}$.

Adapting the last part of the proof in section 2.1, we get the isomorphism: $H^2(\Omega^p) \cong H^{p,2}(X, \mathbb{C})$.

3. Usual proof of the isomorphism.

3.1. Let F be a sheaf of $\mathbb{C}$-vector spaces on a topological space X, on call *résolution of F* an exact sequence of morphisms of sheafs of $\mathbb{C}$-vector spaces

$$(L^*) \qquad 0 \to F \xrightarrow{j} L^0 \xrightarrow{d} L^1 \xrightarrow{d} \ldots \xrightarrow{d} L^n \xrightarrow{d} \ldots$$

Following a demonstration of de Rham's theorem by [W 52] A. Weil, Sur le théorème de de Rham, Comment. Math. Helv., **26** (1952), 119-145,

J.-P. Serre proved:

3.2. **Abstract de Rham's theorem**.- *On a topological space* X*, let* (L^*) *be a resolution of a sheaf* F *such that, for* $m \geq 0$ *and* $q \geq 1$*,* $H^q(X, L^m) = 0$*. Then there exists a canonical isomorphism*

$$H^m(L^*(X)) \to H^m(X, F)$$

where $L^*(X)$ *is the complex*

$$0 \to L^0(X) \to L^1(X) \to \ldots \to L^m(X) \to \ldots$$

of the sections of (L^*).

3.3. d'' **Lemma**.- On an open coordinates neighborhood U (with coordinates $(z_1, \ldots, z_n)$) of a complex analytic manifold, the exterior differential $d = d' + d''$ where $d'' = \sum_{j=1}^{n} \frac{\partial}{\partial \overline{z}_j} d\overline{z}_j$. We have $d''^2 = 0$; this definition is intrinsic. In the same way, on U, every differential form of degree r, $\varphi = \varphi^{r,0} + \ldots, \varphi^{0,r}$ where $\varphi^{u,v} = \sum \varphi_{k_1 \cdots k_u l_1 \cdots l_v} dz_{k_1} \wedge \cdots dz_{k_u} \wedge d\overline{z}_{l_1} \wedge \cdots \wedge d\overline{z}_{l_v}$; the form $\varphi^{u,v}$ of bidegree or type (u, v) is define intrinsically.

Lemma.- *If a germ of differential form C^∞ t is d''-closed, of type (p, q), $q \geq 1$, there exists a germ differential form C^∞ s of type $(p, q-1)$ such that $t = d''s$.*

The Lemma is also valid for *currents* (differential forms with coefficients distributions).

It is proved by P. Dolbeault in the C^ω case, by homotopy, as can been the Poincaré lemma. H. Cartan brings the proof to the C^ω case by a potential theoritical method [Do 53]. Simultanously, the lemma has been proved by A. Grothendieck, by induction on the dimension, from the case $n = 1$ a consequence of the non homogeneous Cauchy formula see [Ca 53], exposé 18).

3.4. A sheaf F on a paracompact space X is said to be *fine* if, for every open set U of a basis of open sets of X and for every closed set $A \subset U$, exists an endomorphism of F equal to the identity at every point of A and to 0 outside U. If F is fine, then $H^q(X,F) = 0$ for every $q \geq 1$.

From d'' Lemma follows the following resolution of the sheaf Ω^p of the holomorphic differential p-formes:

$$0 \rightarrow \Omega^p \xrightarrow{j} \mathcal{E}^{p,0} \xrightarrow{d''} \mathcal{E}^{p,1} \xrightarrow{d''} \dots \xrightarrow{d''} \mathcal{E}^{p,q} \xrightarrow{d''} \dots$$

Recall:

Let $\mathcal{E}^{p,q}$ be the sheaf of differential forms (or currents) of type (p,q) a complex analytic manifold X.

$$
\begin{aligned}
Z^{p,q}(X,\mathbb{C}) \quad &= \mathrm{Ker}(\mathcal{E}^{p,q}(X) \stackrel{d''}{\to} \mathcal{E}^{p,q+1}(X)) \\
B^{p,q}(X,\mathbb{C}) \quad &= \mathrm{Im}(\mathcal{E}^{p,q-1}(X) \stackrel{d''}{\to} \mathcal{E}^{p,q}(X))
\end{aligned}
$$

We call d''-cohomologie group of type (p,q) of X, the $\mathbb{C}$-vector space quotient

$$
H^{p,q}(X,\mathbb{C}) = Z^{p,q}(X,\mathbb{C})/B^{p,q}(X,\mathbb{C})
$$

The sheaf $\mathcal{E}^{p,q}$ is fine as can be seen, using, in the above notations , the endomorphism obtained by multiplication by a function C^∞ with support into U, equal to 1 over A. From the abstract de Rham's Theorem, we get:

Théorème [Do 53a].- *On every paracompact complex analytic manifold X, there exists a canonical isomorphism*

$$H^q(X,\Omega^p) \stackrel{\cong}{\to} H^{p,q}(X,\mathbb{C})$$

This theorem, valid for the cohomology with closed supports when X is paracompact, is also valid for the cohomology with compact supports, i.e. defined by the cochaines with compact supports, if X est locally compact and, more generally, on any complex analytic manifold, for a given family supports.

4. **Closed holomorphic differential forms** [Do 53a],[Do 53b].

1. Fix $p \geq 0$, the sheaf $B^p = \sum_{qr} \mathcal{E}^{p+r,q}$,$q \geq 0$, $r \geq 0$ is graduated by $(p+q+r)$ and stable under d; the same is true for the space $\mathcal{B}^p$ of the sections of B^p; then the space of d-cohomology $H(\mathcal{B}^p)$ is graduated; let $K^{p,q}$ the subspace of the elements of degree $p+q$. The sheaf $\mathcal{E}$ can be replaced by the sheaf of currents.

Let E^p be the sheaf of germs of *closed* holomorphic differential forms of degree p on X. Then, using again the d"-lemma, we get:

Theorem. *For every integers* $p, q \geq 0$*, the* $\mathbb{C}$*-vector space* $H^q(X, E^p)$ *is canonically isomorphic to the* $\mathbb{C}$*-vector space* $K^{p,q}(X, \mathbb{C})$*.*

2. *Remark on the multiplicatve structure of the cohomology*. The exterior product defines a multiplication among the differential forms which is continuous in the topology of sheaves, hence a structure of bigradued algebra for the cohomology.

3. *Relations between the cohomologies H and K.*

Theorem. *The following two exact sequences are isomorphic*:

$$0 \to K^{p,0}(\mathbb{C}) \to H^{p,0}(\mathbb{C}) \to K^{p+1,0}(\mathbb{C}) \to K^{p,1}(\mathbb{C}) \to \dots$$

$$0 \to H^0(E^p) \ \to H^0(\Omega^p) \to H^0(E^{p+1}) \ \to H^1(E^p) \to \ \dots$$

The homorphisms of the first sequence are respectively induced by the projection, the operator d' up to sign and the injection.

The homorphisms of the second sequence are defined by the exact sequence of coefficients

$$0 \to E^p \to \Omega^p \to E^{p+1} \to 0$$

The vertical isomorphisms are those of Theorems 3.4 *and* 4.1.

5. Remarks about Riemann surfaces, algebraic and Kähler manifolds.

1. On a *Riemannian surface*, the complex dimension being 1, all the holomorphic or meromorphic differential forms are d and d"-closed. The fist Betti number is given by the dimension of the spaces of holomorphic forms (first kind) or meromorphic forms of the second kind. One question was: what can be said on complex manifolds of higher dimension?

Recall that closed Riemannian surfaces are the 1-dimensional compact Kähler and open Riemannian surfaces are the 1-dimensional Stein manifolds.

2. *Let X be a compact Kähler manifold, the harmonic operator relative to the Laplacien $\square = d'' \delta'' + \delta'' d''$ defines an isomorphism from $H^{p,q}(X, \mathbb{C})$ onto the $\mathbb{C}$-vector space of harmonic forme of type (p, q).*

In particular, the Hodge decomposition theorem is translated into

$$H^r(X, \mathbb{C}) \cong \bigoplus_{p+q=r;\ p,q \geq 0} H^q(X, \Omega^p)$$

In this way, the cohomology space $H^r(X, \mathbb{C})$ is described by cohomology classes with values in sheaves only depending on the complex analytic structure of the manifold X. The spaces $H^q(X, \Omega^p)$ are a natural generalization of the space $\mathcal{O}(X) = H^0(X, \Omega^0)$ of the holomorphic functions on X.

3. *Let* X *be a Stein manifold,* the sheaves Ω^p being analytic coherent, from Theoreme B on Stein manifolds, we have: $H^q(X, \Omega^p) = 0$ *for* $q \geq 1$*, in other words, the global* d" *problem* d"$g = f$*, always has a solution* g *for a form* f d"*-closed of type* (p, q) *with* $q \geq 1$*.*

6. Frölisher's spectral sequence.

[F 55] A. Frölicher, Relations between the cohomology groups of Dolbeault and topological invariants, Proc. Nat. Ac. Sci. U.S.A., **41** (1955), 641-644.

A spectral sequence is defined which relates the d"-cohomology groups as invariants of the complex structure to the groups of de Rham as topological invariants.

Theorem. *The* d" *-groups* $H^q(X, \Omega^p)$ *form the term* E_1 *of a spectral sequence whose term* E_∞ *is the associated graded* $\mathbb{C}$*-module of the conveniently filtered de Rham groups.The spectral sequence is stationary after a finite number of steps:* $E_\infty = E_N$ *for* N *large enough.*

In the Kähler case, the spectral sequence degenerates at the first step: $E_1^{qp} \cong E_2^{qp} \cong \ldots \cong E_\infty^{qp}$.

Applications.

1. Let $b_{pq} = \text{dim} H^q(X, \Omega^p)$; $\text{dim}_{\mathbb{C}} X = n$

Let χ be the Euler characteristic of X, then:

$$\chi = \sum_{p,q=0}^{n} (-1)^{p+q} b_{pq}$$

2. (r^{th}-Betti number) $\leq \sum_{p+q=r} b_{p,q}$; $r = 0, 1, \ldots, 2n$

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