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About the characterization of some residue currents

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Contents

1. Introduction.

1.1. Residue current in dimension 1. Let $\omega = g(z)dz$ be a meromorphic 1-form on a small enough open set $0 \in U \subset \mathbb{C}$ having 0 as unique pole, with multiplicity k:

$$g = \sum_{l=1}^{k} \frac{a_{-l}}{z^l} + \text{holomorphic function}$$

Note that ω is d-closed.

Let $\psi = \psi_0 d\bar{z} \in \mathcal{D}^1(U)$ be a 1-test form. In general $g\psi$ is not integrable, but the principal value

$$Vp[\omega](\psi) = \lim_{\epsilon \to 0} \int_{|z| \geq \epsilon} \omega \wedge \psi$$

exists, and $dVp[\omega] = d''Vp[\omega] = \text{Res}[\omega]$ is the residue current of ω. For any test function φ on U,

$$\text{Res}[\omega](\varphi) = \lim_{\epsilon \to 0} \int_{|z| = \epsilon} \omega \wedge \varphi$$

Then $\text{Res}[\omega] = 2\pi i \, \text{res}_0(\omega)\delta_0 + dB = \sum_{j=0}^{k-1} b_j \frac{\partial^j}{\partial z^j}\delta_0$ where $\text{res}_0(\omega) = a_{-1}$ is the Cauchy residue. We remark that δ_0 is the integration current on the subvariety $\{0\}$ of U, that $D = \sum_{j=0}^{k-1} b_j \frac{\partial^j}{\partial z^j}$ and that $b_j = \lambda_j a_{-j}$ where the λ_j are universal constants.

Conversely, given the subvariety $\{0\}$ and the differential operator D, then the meromorphic differential form ω is equal to gdz, up to holomorphic form; hence the residue current $\text{Res}[\omega] = D\delta_0$, can be constructed.

1.2. Characterization of holomorphic chains. P. Lelong (1957) proved that a complex analytic subvariety V in a complex analytic manifold X defines an integration current $\varphi \mapsto [V](\varphi) = \int_{\text{Reg}V} \varphi$ on X. More generally, a *holomorphic p-chain* is a current $\sum_{l \in L} n_l [V_l]$ where $n_l \in \mathbb{Z}$, $[V_l]$ is the integration current defined by an irreductible p-dimensional complex analytic subvariety V_l, the family $(V_l)_{l \in L}$ being locally finite.

During more than twenty years, J. King [K 71], Harvey-Shiffman [HS 74], Shiffman [S 83], H. Alexander [A 97] succeeded in proving the following structure theorem: *Holomorphic p-chains on a complex manifold X are exactly the rectifiable d-closed currents of bidimension (p,p) on X.*

In the case of section 1.1, Res $[\omega]$ is the holomorphic chain with complex coefficients $2\pi i\ \mathrm{res}_0(\omega)\delta_0$ if and only if 0 is a simple pole of ω.

1.3. Our aim is to characterize residue currents using rectifiable currents with coefficients that are principal values of meromorphic differential forms and holomorphic differential operators acting on them.

We present a few results in this direction.

The structure theorem of section 1.2 concerns complex analytic varieties and closed currents. So, after generalities on residue currents of semi-meromorphic differential forms, we will concentrate on residue currents of closed meromorphic forms.

2. Preliminaries: local description of a residue current ([D 93], section 6)

2.1. We will consider a finite number of holomorphic functions defined on a small enough open neighborhood U of the origin 0 of $\mathbb{C}^n$, with coordinates $(z_1, \dots, z_n)$. For convenient coordinates, any semi-meromorphic differential form, for U small enough, can be written $\frac{\alpha}{f}$, where $\alpha \in \mathcal{E}^{\cdot}(U))$, $f \in \mathcal{O}(U)$ and

$$f = u_j \prod_k {}_j\rho_k^{r_k},$$

where the ${}_j\rho_k$ are irreducible distinct Weierstrass polynomials in z_j and the $r_k \in \mathbb{N}$ are independent of j, moreover u_j is a unit at 0, i.e., for U small enough, u_j does not vanish on U. Let B_j be the discriminant of the polynomial ${}_j\rho = \prod_k {}_j\rho_k$ and let $Y_k = Z({}_j\rho_k)$; it is clear that Y_k is independent of j. Let $Y = \cup_k Y_k$ and $Z = \mathrm{Sing}\ Y$.

After shrinkage of $(0 \in)\ U$, the following expressions of $\frac{1}{f}$ are valid on U: for every $j \in [1, \dots, n]$,

$$\frac{1}{f} = u_j^{-1} \sum_k \sum_{\mu=1}^{r_k} {}^j c_\mu^k \frac{1}{{}_j\rho_k^\mu}$$

where ${}^j c_\mu^k$ is a meromorphic function whose polar set, in Y_k, is contained in $Z(B_j)$. Notice that B_j is a holomorphic function of $(z_1, \dots, \widehat{z_j}, \dots, z_n)$. In the following, for simplicity, we omit the unit u_j^{-1}.

2.2. Let $\omega = \frac{1}{f}$, $\quad Vp[\omega](\psi) = \lim_{\epsilon \to 0} \int_{[f] \geq \epsilon} \omega \wedge \psi$; $\ \psi \in \mathcal{D}^{n,n}(U)$. The residue of ω is

$$\mathrm{Res}[\omega] = (dVp - Vpd)[\omega] = (d''Vp - Vpd'')[\omega]$$

For every $\varphi \in \mathcal{D}^{n,n-1}(U)$, let $\varphi = \sum_{j=1}^n \varphi_j$ with

$$\varphi_j = \psi_j dz_1 \wedge \dots \wedge d\overline{z}_1 \wedge \dots \wedge \widehat{d\overline{z}_j} \wedge \dots$$

Then, from Herrera-Lieberman [HL 71], and the next lemma about B_j ,
we have:

$$\mathrm{Res}[\omega](\varphi) = \sum_{j=1}^n \sum_k \sum_{\mu=1}^{r_k} \lim_{\delta \to 0} \lim_{\epsilon \to 0} \int_{|B_j| \geq \delta |{}_j\rho_k| = \epsilon} {}^j c_\mu^k \frac{1}{{}_j\rho_k^\mu} \varphi_j.$$

The lemma we have used here is the following:
Lemma 2.1. ([D 93], Lemma 6.2.2).

$$\mathrm{Res}[\omega](\varphi_j) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \int_{|B_j| \geq \delta |f| = \epsilon} \omega \varphi_j.$$

Outside $Z(B_j)$, for $|\ {}_j\rho_k\ |$ small enough (since $\dfrac{\partial_j \rho_k}{\partial z_j} \neq 0$), we take $(z_1, \ldots, z_{j-1}, {}_j\rho_k, z_{j+1}, \ldots, z_n)$ as local coordinates.

2.3. Notations. For the sake of simplicity, until the end of this section, we assume $j = 1$ and write ρ_k, c^k_μ instead of ${}_1\rho_k$, ${}^1c^k_\mu$. Outside $Z(B_1)$, we take $(\rho_k, z_2, \ldots, z_n)$ as local coordinates; then, for every C^∞ function h and every $s \in \mathbb{N}$, we have

$$\frac{\partial^s h}{\partial \rho_k^s} = \frac{1}{(\frac{\partial \rho_k}{\partial z_1})^{2s-1}} D_s h, \quad \text{for } s \geq 1,$$

where $D_s = \sum_{\alpha=1}^{s} \beta^s_\alpha \dfrac{\partial^\alpha}{\partial z_1^\alpha}$, β^s_α is a holomorphic function determined by ρ_k and $D_0 = \Big(\dfrac{\partial \rho_k}{\partial z_1}\Big)^{-1}$.

Let

$$g^\mu_l = \binom{\mu-1}{l} \frac{1}{\Big(\frac{\partial \rho_k}{\partial z_1}\Big)^{2\mu-4}} D_l \Big(\frac{c^k_\mu}{\frac{\partial \rho_k}{\partial z_1}}\Big), (0 \leq l \leq \mu - 2);$$

$$g^\mu_{\mu-1} = \frac{1}{\Big(\frac{\partial \rho_k}{\partial z_1}\Big)^{2\mu-3}} D_{\mu-1} \Big(\frac{c^k_\mu}{\frac{\partial \rho_k}{\partial z_1}}\Big)$$

Let $Vp^1_{Y_k,B_1}[g^\mu_l]$ also denote the direct image, by the inclusion $Y_k \to U$, of the Cauchy principal value $Vp_{Y_k,B_1}[g^\mu_l]$ of $g^\mu_l\ |_{Y_k}$;

$D^{\mu,l}_{1,k} = \sum_{\alpha=1}^{\mu-1-l} (-1)^\alpha \beta^{\mu-1-l}_\alpha \dfrac{\partial^\alpha}{\partial z_1^\alpha}$, and $D^{\mu,\mu-1}_{1,k} = \mathrm{id}$.

2.4. Final expression of the residue. All what has been done for $j = 1$ is valid for any $j \in \{1, \ldots, n\}$: the principal value $Vp^j(k, \mu, l) = Vp\ {}^j_{Y_k,B_j}[g^\mu_l]$ defined on Y_k and the holomorphic differential operator $D^{\mu,l}_{j,k}$. We also denote $Vp^j(k, \mu, l)$ the direct image of the principal value by the canonical injection $Y \hookrightarrow U$. Then, denoting L the inner product, we have:

$$(*) \qquad \mathrm{Res}[\omega](\varphi) = 2\pi i \sum_{j=1}^{n} \Big[\sum_k \sum_{\mu=1}^{r_k} \frac{1}{(\mu-1)!} \sum_{l=0}^{\mu-1} D^{\mu,l}_{j,k} Vp^j(k, \mu, l)\Big] \big(\frac{\partial}{\partial z_j} \mathrm{L} \varphi_j\big)$$

3. The case of simple poles.

3.1. The case $\omega = \dfrac{1}{f}$.

Lemma 3.1. *For a simple pole and for every k, ${}^jc^k_1$ is holomorphic.*

Proof. Let $w = z_j$ and $y = (z_1, \ldots, \hat{z}_j, \ldots, z_n)$. At points $z \in U$ where $B_j(z) \neq 0$, for given y, let $w_{ks}, s = 1, \ldots, s_k$, be the zeros of ρ_k. For given y, $\rho_k = \prod_{s=1}^{s_k} (w - w_{ks})$,

$$\frac{1}{f} = u_j \sum_k \sum_{s=1}^{s_k} {}^j\mathcal{C}^{k,s}_1 (w - w_{ks})^{-1}$$

where ${}^j\mathcal{C}^{k,s}_1 = \dfrac{1}{\frac{\partial}{\partial w} f(w_{ks}, y)}$; let $\prod^s_\sigma$ denote the product for all $\sigma \neq s$,

$$\sum_{s=1}^{s_k} {}^j\mathcal{C}^{k,s}_1 (w - w_{ks})^{-1} = \sum_{s=1}^{s_k} {}^j\mathcal{C}^{k,s}_1 \frac{\prod^s_\sigma (w - w_{k\sigma})}{\prod_\sigma (w - w_{k\sigma})} = {}^jc^k_1(w, y) \rho_k^{-1},$$

with

$${}^{j}c_1^k(w,y) = \sum_{s=1}^{s_k} \frac{\prod_\sigma^s (w - w_{k\sigma})}{\frac{\partial}{\partial w} f(w_{ks}, y)} \text{ ([D 57], IV.B.3 et C.1).}$$

Here ${}^{j}c_1^k(w,y)$ holomorphically extends to points of U where the w_s are not all distinct because: if w_s appears m times in $\prod_\sigma (w - w_{k\sigma})$, it appears $(m-1)$ times in the numerator and the denominator of $\frac{\prod_\sigma^s (w - w_{k\sigma})}{\frac{\partial}{\partial w} f(w_{ks}, y)}$.
□

All the poles of ω are simple, i.e. for every k, $r_k = 1$; then $\mu = 1$, $l = 0$.

$$\mathrm{Res}[\omega](\varphi) = 2\pi i \sum_{j=1}^{n} \Big[\sum_k D_{j,k}^{1,0} Vp^j(k,1,0)\Big]\big(\frac{\partial}{\partial z_j} \mathrm{L}\varphi_j\big)$$

$$\text{But } D_{1,k}^{1,0} = \mathrm{id};\ D_0 = \Big(\frac{\partial \rho_k}{\partial z_1}\Big)^{-1};\ g_{\mu-1}^{\mu} = \frac{1}{\Big(\frac{\partial \rho_k}{\partial z_1}\Big)^{2\mu-3}} D_{\mu-1}\Big(\frac{c_\mu^k}{\frac{\partial \rho_k}{\partial z_1}}\Big);\ g_0^1 = \frac{1}{\Big(\frac{\partial \rho_k}{\partial z_1}\Big)^{-1}} D_0\Big(\frac{c_1^k}{\frac{\partial \rho_k}{\partial z_1}}\Big)$$

$$= \frac{1}{\Big(\frac{\partial \rho_k}{\partial z_1}\Big)^{-1}} \Big(\frac{\partial \rho_k}{\partial z_1}\Big)^{-1} \Big(\frac{c_1^k}{\frac{\partial \rho_k}{\partial z_1}}\Big) = \Big(\frac{\partial \rho_k}{\partial z_1}\Big)^{-1} c_1^k;$$

$$Vp^j(k,1,0) = Vp\ {}^{j}_{Y_k,B_j}[g_0^1] = Vp\ {}^{j}_{Y_k,B_j}\Big[\Big(\frac{\partial \rho_k}{\partial z_j}\Big)^{-1}\ {}^{j}c_1^k\Big],$$

hence

$$\mathrm{Res}[\omega](\varphi) = 2\pi i \sum_{j=1}^{n} \Big[\sum_k Vp\ {}^{j}_{Y_k,B_j}\big[\Big(\frac{\partial \rho_k}{\partial z_j}\Big)^{-1}\ {}^{j}c_1^k\big]\big(\frac{\partial}{\partial z_j} \mathrm{L}\varphi_j\big)\Big]$$

where ${}^{j}c_1^k$ is holomorphic.

3.2. The case of any degree. Let $\omega = \frac{\alpha}{f}$. Then Res $[\omega] = \alpha \wedge$ Res$(\frac{1}{f})$. Moreover, d Res $[\omega] = \pm\mathrm{Res}[d\omega]$, then Res $[\omega]$ is d-closed if ω is d-closed.

4. Expression of the residue current of a closed meromorphic differential form.

In this section and a part of the following one, we give statements on residue currents according to the general hypotheses and proofs of sections 2 and 3. Proofs in a particular case where the polar set is equisingular and the singularity of the polar set is a 2-codimensional smooth submanifold are given in ([D 57], IV.D).

4.1. Closed meromorphic differential forms.

4.1.1. Let $\omega = \frac{\alpha}{f}$ be a d-closed meromorphic differential p-form on a small enough open neighborhood U of the origin 0 of $\mathbb{C}^n$. From section 2.1, we get $\omega = \sum \omega_k$ with $\omega_k = \sum_{\mu=1}^{r_k} {}^{j}c_\mu^k \frac{\alpha}{{}_j\rho_k^\mu}$ for every $j = 1, \ldots, n$. We have

$${}^{j}c_\mu^k = \frac{{}^{j}a_\mu^k(z_1, \ldots, z_n)}{{}^{j}b_\mu^k(z_1, \ldots, \widehat{z_j}, \ldots, z_n)},$$

where a and b are holomorphic. Then $d\omega = \sum d\omega_k$ and $d\omega_k$ is the quotient of a holomorphic form by a product of ${}^{j}b_\mu^k(z_1, \ldots, \widehat{z_j}, \ldots, z_n)$ and ${}_j\rho_k^{r_k+1}$ (see [D 57], IV,D.1).

As at the end of section 2.2, using the local coordinates

$$(z_1, \ldots, z_{j-1}, \rho_k, z_{j+1}, \ldots, z_n),$$

we have

$$\omega_k = \sum_{\mu=1}^{r_k} [{}_jA_\mu^k \wedge\ {}_j\rho_k^{-\mu} d_j\rho_k +_j \rho_k^{-\mu} B_k'], \tag{4.1}$$

where the coefficients are meromorphic.

Let $\mathcal{R}_j$ be the ring of meromorphic forms on U whose coefficients are quotients of holomorphic forms on U by products of powers of $\dfrac{\partial_j \rho_k}{\partial z_j}$ and ${}^j b^k_\mu$.

Lemma 4.1 ([D 57], Lemme 4.10). *Assume that $d\omega_k \in \mathcal{R}_j$. Then*

$$\omega_k =_j \rho_k^{-1} d_j\rho_k \wedge a^k_j + \beta^k_j + dR^k_j$$

with

$$R^k_j = \sum_{\nu=1}^{r_k-1} {}_j e^k_{\nu j} \rho_k^{-\nu} \ \text{ and } \ da^k_j = d_j\rho_k \wedge {}^k a'_j + C^k_{j\,j}\rho_k,$$

where $a^k_j, \beta^k_j, \ {}_j e^k_\nu, \ {}^k a'_j, C_j \in \mathcal{R}_j$ and are independent of dz_j.

4.1.2. Let φ be of type $(n-p, n-1)$. Then

$$\varphi = \sum \varphi_j, \text{with } \ \varphi_j = \sum \psi_{l_1,\ldots,l_{n-p}} dz_{l_1} \wedge \ldots \wedge dz_{l_{n-p}} \wedge \ldots \wedge \widehat{d\overline{z}_j} \wedge \ldots$$

Proposition 4.2. *Let $\omega = \dfrac{\alpha}{f}$ be a d-closed meromorphic p-form on U. Given a coordinate system on U, and with notations of section 2.1, there exists a current $S_j^{p-1,1}$ such that $d''S_j|_{U\setminus Z} = 0$, $\ \operatorname{supp} S_j = Y$ and, for every k, j, a d-closed meromorphic $(p-1)$-form A^k_j on Y_k with polar set Z such that*

$$Res[\omega](\varphi) = \sum_{j=1}^{n} \Big(2\pi i \sum_k Vp_{Y_k,B_j} A^k_j + d'S_j\Big)\Big(\frac{\partial}{\partial z_j} \mathrm{L} \varphi_j\Big).$$

When the coordinate system is changed, the first term of the parenthesis is modified by addition of $2\pi i \sum_k d' Vp_{Y_k,B_j}[F^k_j]$ where F^k_j is a meromorphic $(p-2)$-form on Y_k with polar set Z.

Here $2\pi i \sum_{j=1}^n \sum_k Vp_{Y_k,B_j} A^k_j(\cdot_j)$ will be called the *reduced residue* of ω.

Proof. Apply the proof of (*) (section 2) to the meromorphic form of Lemma 4.1.

We shall use the expression of $\mathrm{Res}[\omega](\varphi)$ of section 2.2, for ω closed.

For k and j fixed, we consider

$$J_{kj} = \lim_{\delta\to 0}\lim_{\epsilon\to 0} \int_{|B_j|\ge\delta, |_j\rho_k| = \epsilon} \omega_k(\varphi_j).$$

Then $\mathrm{Res}[\omega](\varphi) = \sum_{k,j} J_{kj}$.

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} \int_{|B_j|\ge\delta, |_j\rho_k| = \epsilon} dR^k_j \wedge \varphi_j = (-1)^p \lim_{\delta\to 0}\lim_{\epsilon\to 0} \int_{|B_j|\ge\delta, |_j\rho_k| = \epsilon} R^k_j \wedge d\varphi_j.$$

Let S^k_j be the current defined by

$$S^k_j(\psi_j) = -\lim_{\delta\to 0}\lim_{\epsilon\to 0} \int_{|B_j|\ge\delta, |_j\rho_k| = \epsilon} R^k_j \wedge \psi_j.$$

By Lemme 4.1. R^k_j is independent of dz_j.

Let $\psi_j = dz_j \wedge \eta^j + \xi^j$, where ξ^j is independent of dz_j, then $\eta^j = \frac{\partial}{\partial z_j} \mathrm{L} \psi_j$.

After change of coordinates:

$$S^k_j(\psi_j) = -\lim_{\delta\to 0}\lim_{\epsilon\to 0} \int_{|B_j|\ge\delta, |_j\rho_k| = \epsilon} \Big(\frac{\partial_j\rho_k}{\partial z_j}\Big)^{-1} R^k_j \wedge d_j\rho_k \wedge \eta^j \tag{4.2}$$

$$= (-1)^p 2\pi i \lim_{\delta\to 0} \sum_{\nu} \int_{Y_k |B_j|\geq\delta} (\nu-1)!^{-1} \Big(\frac{\partial^{\nu-1}\big({}_j e^k_\nu \wedge \eta^j \big(\frac{\partial_j \rho_k}{\partial z_j}\big)^{-1}\big)}{\partial_j {\rho_k}^{\nu-1}} \Big)_{{}_j\rho_k=0}$$

We have $S_j(\psi_j) = \sum_k S^k_j$.

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} \int_{|B_j|\geq\delta, |{}_j\rho_k|=\epsilon} {}_j{\rho_k}^{-1} d_j\rho_k \wedge a^k_j + \beta^k_j = 2\pi i \lim_{\delta\to 0} \int_{|B_j|\geq\delta} a^k_j\ |_{Y_k} = 2\pi i V p_{Y_k, B_j} A^k_j, \text{ with } A^k_j = a^k_j\ |_{Y_k}$$

The last alinea is proved as in ([D 57], IV.D.4). □

Corollary 4.3. *The current S_j is obtained by application of holomorphic differential operators to currents principal values of meromorphic forms supported by the irreducible components of Y.*

Proof. The corollary follows from the above expression for S_j and the computations in section 2. □

We remark that d' itself is a holomorphic differential operator.

4.2. Particular cases.

4.2.1. The case $p = 1$. With the notations of Proposition 4.2, the forms A^k are of degree 0 and are d-closed, hence constant and unique: the reduced residue is a *divisor with complex coefficients.*

4.2.2. With the hypotheses and the notations of section 2.1, if all the multiplicities r_k are equal to 1, the reduced residue is uniquely determined and the current $S = 0$.

4.3. Comparison with the expression of $Res[\omega]$ in section 2, when ω is d-closed.

The reduced residue is equal to

$$2\pi i \sum_{j=1}^{n} \Big[\sum_k Vp\ {}^{j}_{Y_k, B_j} [\Big(\frac{\partial\rho_k}{\partial z_j}\Big)^{-1}\ {}^j c^k_1] (\frac{\partial}{\partial z_j} \mathrm{L}(\alpha\wedge .)_j) \Big].$$

It is well defined if all the poles of ω are simple.

5. Generalization of a theorem of Picard. Structure of residue currents of closed meromorphic forms.

5.1. The theorem of Picard [P 01] characterizes the divisor with complex coefficients associated to a d-closed differential form, of degree 1 of the third kind, on a complex projective algebraic surface; this result has been generalized by S. Lefschetz (1924): "the divisor has to be homologous to 0", then by A. Weil (1947). Locally, one of its assertions is a particular case of the theorem of Dickenstein-Sessa ([DS 85], Theorem 7.1): *Analytic cycles are locally residual currents* (see section 5.5), with a variant by D. Boudiaf ([B 92], Ch.1, sect.3).

5.2. Main results.

Theorem 5.1. *Let X be a complex manifold which is compact Kähler or Stein, and Y be a complex hypersurface of X, then $Y = \cup_\nu Y_\nu$ is a locally finite union of irreducible hypersurfaces. Let $Z =$ Sing Y, and let A_ν be a d-closed meromorphic $(p-1)$-form on Y_ν with polar set $Y_\nu \cap Z$ such that the current $t = 2\pi i \sum_\nu V p_{Y_\nu} A_\nu$ is d-closed.*

Then the following two conditions are equivalent:

(i) *t is the residue current of a d-closed meromorphic p-form on X having Y as polar set with multiplicity one.*

(ii) *$t = dv$ on X, where v is a current, i.e., is cohomologous to 0 on X.*

Proof. From section 4 locally, and a sheaf cohomology machinery globally; detailed proof will be given later for the more general theorem 5.5. □

For $p = 1$, the A_ν are complex constants, then t is the divisor with complex coefficients $2\pi i \sum_\nu A_\nu Y_\nu$.

Corollary 5.1.1. *Under the hypotheses of Theorem 5.1, every residue current of a closed meromorphic p-form appears as a divisor, homologous to 0, whose coefficients are principal values of meromorphic $(p-1)$-forms on the irreducible components of the support of the divisor and conversely.*

Let $\mathcal{R}^{loc}_{q,q}(X)$ be the vector space of locally rectifiable currents of bidimension (q,q) on the complex manifold X and

$$\mathcal{R}^{loc\mathbb{C}}_{q,q}(X) = \mathcal{R}^{loc}_{q,q} \otimes_{\mathbb{Z}} \mathbb{C}(X)$$

.

Theorem 5.2. *Let $T \in \mathcal{R}^{loc\mathbb{C}}_{q,q}(X)$, $dT = 0$. Then T is a holomorphic q-chain with complex coefficients.*

This is the structure theorem of holomorphic chains of Harvey-Shiffman-Alexander for complex coefficients; thanks to it, divisors will be translated into rectifiable currents.

Theorem 5.3. *Let X be a Stein manifold or a compact Kähler manifold. Then the following conditions are equivalent:*

(i) T is the residue current of a d-closed meromorphic 1*-form on X having* supp T *as polar set with multiplicity* 1*;*

(ii) $T \in \mathcal{R}^{loc\mathbb{C}}_{n-1,n-1}(X)$, $T = dV$.

In the same way, we can reformulate the Theorem 5.1 with rectifiable currents:

Theorem 5.4. *Let X be a Stein manifold or a compact Kähler manifold. Then the following conditions are equivalent:*

(i) $T = \sum_\nu a_\nu T_\nu$, with $T_\nu \in \mathcal{R}^{loc\mathbb{C}}_{n-1,n-1}(X)$, d-closed, and a_ν the principal value of a d-closed meromorphic $(p-1)$-form on supp T_ν*, such that $T = dV$;*

(ii) T is the residue current of a d-closed meromorphic p-form on X having $\cup_l T_l$ as polar set with multiplicity 1.

5.3. Remark. The global Theorem 5.1 gives also local results since any open ball centered at 0 in $\mathbb{C}^n$ is a Stein manifold.

5.4. Generalization.

5.4.1. With the notations of section 4.1, what has been done with the current $2\pi i \sum_\nu Vp_{Y_\nu} A_\nu$ is also possible in the general case. The current S is defined as follows: let $\psi = \sum_j \psi_j$, then $S(\psi) = \sum_j \sum_k S^k_j(\psi_j)$.

From (4.2), we have:

$$S^k_j(\psi_j) = 2\pi i \sum_{\mu=1}^{r_k} \sum_{l=0}^{\mu-1} \Delta^{\mu,l}_{j,k} Vp^j_{Y_k,B_j}[\gamma^{\mu j}_{k,l}]\left(\frac{\partial}{\partial z_j} \mathrm{L}\psi_j\right) \tag{5.3}$$

where $\gamma^{\mu j}_{k,l}$ is a meromorphic form on Y_k, with polar set contained in $Y_k \cap \{B_j = 0\}$, and where $\Delta^{\mu,l}_{j,k}$ is a holomorphic differential operator in the neighborhood of Y_k. In the global case, for $Y = \cup_\nu Y_\nu$ locally finite, we take $k = \nu$, the sum $\sum_\nu S^\nu_j$ being locally finite.

Then we will get generalizations of the results in sections 5.2 and 5.3 completing the programme of section 1.3.

Lemma 5.1. *Let m^p be the sheaf of closed meromorphic differential forms. Let $\overline{m}^p$ be the image by* Vp *of m^p in the sheaf of germs of currents on X. Then, for X Stein or compact Kähler manifold, we have the commutative diagram*

$$\begin{array}{ccccccc} H^0(X, m^p) & \to & H^0(X, \overline{m}^p) & \to & H^0(X, \overline{m}^p/E^p) & \to & H^1(X, E^p) \\ & & & & \text{Res}\downarrow & & \downarrow \\ & & & & H^0(X, d''\overline{m}^p) & \to & H^{p+1}(X, \mathbb{C}) \end{array}$$

(from [D 57], IV.D.7)

5.4.2. The residue current of a d-closed meromorphic p-form is globally written $t = 2\pi i \sum_\nu Vp_{Y_\nu} A_\nu + d'S$, where $S = \sum_\nu \sum_j S^\nu_j$, with $dt = 0$, from the local Proposition 4.2.

Theorem 5.5. *If X is a complex manifold which is compact Kähler, or Stein, and Y is a complex hypersurface of X, then $Y = \cup_\nu Y_\nu$ is a locally finite union of irreducible hypersurfaces. Let $Z=$* Sing$Y$*; for every ν, let A_ν be a d-closed meromorphic $(p-1)$-form on Y_ν, and, in the notations of* (5.3) *with $k = \nu$, $\gamma^{\mu j}_{\nu,l}$ be meromorphic $(p-2)$-forms on Y_ν, with polar set $Y_\nu \cap Z$ such that the current $t = 2\pi i \sum_\nu Vp_{Y_\nu} A_\nu + d'S$, with $S = \sum_\nu \sum_j S^\nu_j$, be d-closed.*

Then the following two conditions are equivalent:

(i) t is the residue current of a d-closed meromorphic p-form on X having Y as polar set.

(ii) $t = dv$ on X, where v is a current, i.e. t is cohomologous to 0 on X.

Proof.

$(i) \Rightarrow (ii)$: From Lemma 5.1, the cohomology class of a residue current is 0; it is the case of t.

$(ii) \Rightarrow (i)$: $t = dv$ on X; t of type $(p, 1)$ implies: $t = dv = d''v$; v of type $(p, 0)$; the current v is closed on $X \setminus Y$, therefore it a holomorphic p-form on $X \setminus Y$. Let m^p_Y be the sheaf of closed meromorphic p-forms with polar set Y; the Lemma 5.1 is valid for m^p_Y instead of m^p. At a point $O \in Y$, Y is defined by $\Pi_k \rho_k = 0$ (omitting the index j); the r_k being the integers in (5.3), then $d(\Pi_k \rho_k^{r_k} v) = \Pi_k \rho_k^{r_k} d''v = \Pi_k \rho_k^{r_k} t = 0$ from Lemma 4.1; therefore $\Pi_k \rho_k^{r_k} v$ is a germ of holomorphic form at O and v extends a closed meromorphic form $G \in H^0(X, m^p_Y)$ on X.

We will show that t is the residue current of G.

From Proposition 4.2,

$$\mathrm{Res}[G] = d''\mathrm{Vp}\ G = 2\pi i \sum_\nu \mathrm{Vp}_{Y_\nu} B_\nu + d'T$$

where B_ν and T are of the same nature as A_ν and S.

Lemma 5.2. $M = v -$ Vp G *satisfies* $d''M = 0$.

Proof. We have:

$$d''M = 2\pi i \sum_\nu Vp_{Y_\nu}(A_\nu - B_\nu) + d'(S - T) \tag{5.4}$$

Let O_1 be a non singuler point of Y; there exists k such that: $O_1 \in \{{}_j\rho_k = 0\}$, $(j = 1, \ldots, n)$; in the neighborhood of O_1, ${}_j\rho_k$ can be used as local coordinate. We have: $M = M_j$ where M_j is written with the local coordinates $(\ldots, z_{j-1}, {}_j\rho_k, z_{j+1}, \ldots)$; $d''M = d''M_j$; the support of $d''M$ is Y, then, in the neighborhood of O_1, $d''M_j$ vanishes on the differential forms containing $d_j\rho_k$ or $d_j\overline{\rho}_k$. Then

$$d''M_j = d_j\rho_k \wedge d_j\overline{\rho}_k \wedge N_j \tag{5.5}$$

M_j is of type $(p, 0)$, therefore without term in $d_j\overline{\rho}_k$ and in $d\overline{z}_l$, $l \neq j$.

From (5.5), $\dfrac{\partial M_j}{\partial \overline{z}_l} = 0$, then

$$d''M_j = d_j\overline{\rho}_k \wedge \frac{\partial M_j}{\partial_j\overline{\rho}_k} \tag{5.6}$$

$d''M_j$ is a differential form with distribution coefficients supported by Y_k, therefore, outside Z, from the structure theorem of distributions supported by a submanifold ([Sc 50], ch. III, théorème XXXVII), and from (5.6), the coefficients of $d''M_j$ being those of $\dfrac{\partial M_j}{\partial_j\overline{\rho}_k}$, then $d''M_j$ contains transversal derivatives with respect ${}_j\rho_k$ or ${}_j\overline{\rho}_k$ of order at least equal to $r_k + 1$, what is incompatible with the initial expression (5.4) of $d''M_j$, except if $d''M_j = 0$ outside Z. From (5.4) the $\mathrm{Vp}_{Y_k}(A_\nu - B_\nu)$ and $(S - T)$ being defined as limits of integrals of forms vanishing on $Y \setminus Z$, we have: $d''M = 0$ on X. □

From Lemma 5.2, $\mathrm{Res}[G] = d''v = t$. □

Corollary 5.5.1. *Under the hypotheses of Theorem 5.5, the current S is a sum of currents obtained by application of holomorphic differential operators to principal values of meromorphic forms on the irreducible components Y_ν of Y.*

Corollary 5.5.2. *Under the hypotheses of Theorem 5.5, the residue current of a d-closed meromorphic differential p-form is the sum, cohomologous to 0, of currents obtained by application of holomorphic differential operators to currents $\sum_\nu a_\nu T_\nu$, with $T_\nu \in \mathcal{R}^{loc\mathbb{C}}_{n-1,n-1}(X)$, d-closed, and a_ν the principal value of a meromorphic $(p-1)$-form on* supp T_ν.

5.5. Remarks. The Theorems of the sections 5.2 and 5.4 and their Corollaries are valid for locally residue currents in the terminology of [DS 85]. Results are also valid for any complex analytic manifold, using less natural cohomology (cf [D 57], IV.D.7).

6. Remarks about residual currents [CH 78], [DS 85]**.**

In the classical definition and notations, we consider residual currents $R^p[\mu] = R^pP^0[\mu]$, where μ is a semi-meromorphic form $\frac{\alpha}{f_1.....f_p}$, and α a differential $(p, 0)$-form. Then, $R^p[\mu]$ satisfies a formula analogous to (*) of section 2.4. ([D 93] , section 8).

Locally, one of the assertions of the theorem of Picard is valid for any p, from the result of Dickenstein-Sessa quoted in section 5.1. So generalizations of theorems in sections 5.2 to 5.4, for residual currents, seem valid.

References

[A 97] H. Alexander, Holomorphic chains and the support hypothesis conjecture, *J. of the Amer. Math. Soc.*, **10** (1997), 123-138.

[B 92] D. Boudiaf, *Thèse de l'Université Paris VI*, (1992).

[CH 78] H. Coleff et M. Herrera, Les courants résiduels associés à une forme méromorphe, *Springer Lecture Notes in Math.* **633** (1978).

[DS 85] A. Dickenstein and C. Sessa, Canonical reprentatives in moderate cohomology, *Inv. Math.* **80**, 417-434 (1985).

[D 57] P. Dolbeault, Formes différentielles et cohomologie sur une variété analytique complexe, II, *Ann. of Math.* **65** (1957), 282-330.

[D 93] P. Dolbeault, On the structure of residual currents, *Several complex variables, Proceedings of the Mittag-Leffler Institute, 1987-1988*, Princeton Math. Notes **38** (1993), 258-273.

[HS 74] R. Harvey and B. Shiffman, A characterization of holomorphic chains, *Ann. of Math.* **99** (1974), 553-587.

[HL 71] M. Herrera and D. Lieberman, Residues and principal values on complex spaces, *Math. Ann.* **194** (1971), 259-294.

[K 71] J. King, The currents defined by analytic varieties, *Acta Math.* **127** (1971), 185-220.

[P 01] E. Picard, Sur les intégrales des différentielles totales de troisième espèce dans la théorie des surfaces algébriques, *Ann. Sc. E.N.S.* **18** (1901), 397-420.

[Sc 50], L. Schwartz, Théorie des distributions, new edition, Hermann, Paris 1966.

[S 83] B. Shiffman, Complete characterization of holomorphic chains of codimension one, *Math. Ann.* **274** (1986), 233-256.

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