ON A NONCOMMUTATIVE ALGEBRAIC GEOMETRY

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Abstract. Several sets of quaternionic functions are described and studied with respect to hyperholomorphy, addition and (non commutative) multiplication, on open sets of $\mathbb{H}$, then Hamilton 4-manifolds analogous to Riemann surfaces, for $\mathbb{H}$ instead of $\mathbb{C}$, are defined, and so begin to describe a class of four dimensional manifolds.

Contents

1. Introduction. We first recall the definition of the field $\mathbb{H}$ of quaternions using pairs of complex numbers and a modified Cauchy-Fueter operator (section 2) that have been introduced by C. Colombo and al., [CLSSS07]. We will only use right multiplication. We will consider C^∞ $\mathbb{H}$-valued quaternionic functions defined on an open set U of $\mathbb{H}$ whose behavior mimics the behavior of holomorphic functions on an open set of $\mathbb{C}$. If such a function does not vanish identically, it has an (algebraic) inverse. Finally we describe properties of Hyperholomorphic functions with respect to addition and multiplication.

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In section 3, we characterize the quaternionic functions which are, almost everywhere, hyperholomorphic and whose inverses are hyperlomorphic almost everywhere, on U, as the solutions of a system of two non linear PDE. We find non trivial examples of a solution showing that the considered space of functions is significant; we will call these functions Hypermeromorphic.

At the moment, I am unable to get the general solution of the the system of PDE. Same difficulty for subsequent occurring systems of PDE.

In section 4, we describe a subspace of hyperholomorphic and hypermeromorphic functions defined almost everywhere on U, having "good properties for addition and multiplication"; we again obtain systems of non linear PDE.

In section 5 and the following, we consider globalization of the above notions, define Hamilton 4-manifolds analogous to Riemann surfaces, for $\mathbb{H}$ instead of $\mathbb{C}$, and give examples of such manifolds; our ultimate aim is to describe a class of 4-dimensional manifolds.

2. Quaternions. $\mathbb{H}$-valued functions. Hyperholomorphic functions.

See [CSSS04, CLSSS07, D13].

2.1. Quaternions. If $q \in \mathbb{H}$, then $q = z_1 + z_2\mathbf{j}$ where $z_1, z_2 \in \mathbb{C}$. We have $z_1\mathbf{j} = \mathbf{j}\overline{z}_1$, and note $|q| = |z_1|^2 + |z_2|^2$.

The conjugate of q is $\overline{q} = \overline{z}_1 - z_2\mathbf{j}$. Let us denote $*$ the (right) multiplication in $\mathbb{H}$, then the right inverse of q is: $q^{-1} = |q|^{-1}\overline{q}$

2.2. Quaternionic functions. Let U be an open set of $\mathbb{H} \cong \mathbb{C}^2$ and $f \in C^\infty(U, \mathbb{H})$, then $f = f_1 + f_2\mathbf{j}$, where $f_1, f_2 \in C^\infty(U, \mathbb{C})$. The complex valued functions f_1, f_2 will be called the *components* of f.

2.3. Definitions. Let U be an open neighborhood of 0 in $\mathbb{H} \cong \mathbb{C}^2$.

(a) From now on, we will consider the quaternionic functions $f = f_1 + f_2\mathbf{j}$ having the following properties:

(i) When f_1 and f_2 are not holomorphic, the set $Z(f_1) \cap Z(f_2)$ is discrete on U;

(ii) for every $q \in Z(f_1) \cap Z(f_2)$, J_q^α (.) denoting the *jet of order α at q* (see [M66]), let $m_i = \sup_{\alpha_i} J_q^{\alpha_i}(f_i) = 0$; $m_i, i = 1, 2$, is finite.

$m_q = inf m_i$ is the *order of the zeroe q of f*.

(b) We will also consider the quaternionic functions defined almost everywhere on U (i.e. outside a locally finite set of C^∞ hypersurfaces, namely $Z(f_1), Z(f_2)$).

2.4. Modified Cauchy-Fueter operator $\mathcal{D}$. Hyperholomorphic functions.

See [CLSSS07, F39].

For $f \in C^\infty(U, \mathbb{H})$, with $f = f_1 + f_2\mathbf{j}$,

$\mathcal{D}f(q) = \frac{1}{2}(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2})f(q)$.

A function $f \in C^\infty(U, \mathbb{H})$ is said *hyperholomorphic* if $\mathcal{D}f = 0$.

Characterization of the hyperholomorphic function f on U:

$$\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2} = 0; \ \frac{\partial f_1}{\partial \overline{z}_2} + \frac{\partial \overline{f}_2}{\partial z_1} = 0, \text{ on } U. \tag{1}$$

2.5. Several families of meromorphic functions. The conditions f_1 is holomorphic and f_2 is holomorphic are equivalent on U; the same is true for almost everywhere defined holomorphic functions on U.

By definition, *holomorphic (almost everywhere defined functions of two complex variables* on U are such that $f_2 = 0$, and f_1 is (almost everywhere) holomorphic.

2.5.1. *Consider the almost everywhere defined hyperholomorphic functions on U whose components are real.*

$$f = f_1 + f_2\mathbf{j}$$

According to a remark of Guy Roos in March 2013, they are almost everywhere holomorphic [R13].

2.5.2. *The above considered almost everywhere holomorphic functions are meromorphic and constitute two $\mathbb{H}$-commutative algebras A_1, A_2, with common origin* 0. Let $f = a+b\mathbf{i}$, and $g = c+d\mathbf{j}$, with $a, b, c, d \in \mathbb{R}$ be two almost everywhere defined holomorphic functions i.e.meromorphic functions on U.

A_1 is the set of the meromorphic functions $f = a+b\mathbf{i}$, and A_2 is the set of meromorphic functions $g = c + d\mathbf{j}$, with $a, b, c, d \in \mathbb{R}$

The sums $f + g = a + c + d\mathbf{j} + b\mathbf{i}$ constitute the algebra $A_1 + A_2$ of meromorphic functions.

More generally, $A_{\alpha,\beta} = \alpha A_1 + \beta A_2$, with $\alpha, \beta \in \mathbb{R}$ is an algebra of meromorphic functions on U.

$$A_{\alpha\beta} = \sum_{a,b,c,d,\alpha,\beta\in\mathbb{R}} \alpha(a + b\mathbf{i}) + \beta(c + d\mathbf{j})$$

2.5.3. *We now begin to introduce multiplication for hyperholomorphic functions, addition and scalar muliplication being obvious.*

2.6. Multiplication of almost everywhere defined hyperholomorphic functions.

Proposition 2.1. *Let f', f'' be two almost everywhere defined hyperholomorphic functions. Then, their product $f' * f''$ satisfies:*

$$\mathcal{D}(f' * f'') = \mathcal{D}f' * \mathbf{j}f'' + \big(f'(\frac{\partial}{\partial\bar{z}_1}) + \overline{f}'\mathbf{j}\frac{\partial}{\partial\bar{z}_2}\big)f''$$

Proof. $f' = f_1' + f_2'\mathbf{j}$, $f'' = f_1'' + f_2''\mathbf{j}$ be two hyperholomorphic functions.

We have: $f' * f'' = (f_1' + f_2'\mathbf{j})(f_1'' + f_2''\mathbf{j}) = f_1' f_1'' - f_2'\overline{f}_2'' + (f_1' f_2'' + f_2'\overline{f}_1'')\mathbf{j}$

Compute

$$\frac{1}{2}(\frac{\partial}{\partial\bar{z}_1} + \mathbf{j}\frac{\partial}{\partial\bar{z}_2})(f_1' f_1'' - f_2'\overline{f}_2'' + (f_1' f_2'' + f_2'\overline{f}_1'')\mathbf{j})$$

By derivation of the first factors of the sum $f' * f''$, we get the first term:

$$\frac{1}{2}(\frac{\partial f_1'}{\partial\bar{z}_1} + \mathbf{j}\frac{\partial f_1'}{\partial\bar{z}_2})(f_1'' + f_2''\mathbf{j}) + \frac{1}{2}(\frac{\partial f_2'}{\partial\bar{z}_1} + \mathbf{j}\frac{\partial f_2'}{\partial\bar{z}_2})\mathbf{jj}(\overline{f}_2'' - \overline{f}_1''\mathbf{j})$$

$$= \frac{1}{2}(\frac{\partial f_1'}{\partial\bar{z}_1} + \mathbf{j}\frac{\partial f_1'}{\partial\bar{z}_2})(f_1'' + f_2''\mathbf{j}) + \frac{1}{2}(\frac{\partial f_2'\mathbf{j}}{\partial\bar{z}_1} + \mathbf{j}\frac{\partial f_2'\mathbf{j}}{\partial\bar{z}_2})\mathbf{j}(f_2''\mathbf{j} + f_1'') = \mathcal{D}f' * \mathbf{j}f''$$

By derivation in

$$\frac{1}{2}\Big(\frac{\partial}{\partial \overline{z}_1}+\mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)\big(f_1'f_1''+f_2'\mathbf{j}f_2''\mathbf{j}+(f_1'f_2''\mathbf{j}+f_2'\mathbf{j}f_1'')\big)$$

of the second factors of the sum $f'*f''$, we get the second term (up to factor $\frac{1}{2}$):

$$\begin{aligned}
&f_1'\frac{\partial f_1''}{\partial \overline{z}_1}+\overline{f}_1'\mathbf{j}\frac{\partial f_1''}{\partial \overline{z}_2}+f_1'\frac{\partial f_2''}{\partial \overline{z}_1}\mathbf{j}+\overline{f}_1'\mathbf{j}\frac{\partial f_2''}{\partial \overline{z}_2}\mathbf{j}\\
&\qquad+f_2'\mathbf{j}\frac{\partial f_2''}{\partial \overline{z}_1}\mathbf{j}+\overline{f}_2'\mathbf{j}\frac{\partial f_2''}{\partial \overline{z}_2}+f_2'\mathbf{j}\frac{\partial f_1''}{\partial \overline{z}_1}+\overline{f}_2'\mathbf{j}\mathbf{j}\frac{\partial f_1''}{\partial \overline{z}_2}\\
&=(f_1'+f_2'\mathbf{j})\Big(\frac{\partial}{\partial \overline{z}_1}\Big)(f_1''+f_2''\mathbf{j})+(\overline{f}_1'+\overline{f}_2'\mathbf{j})\mathbf{j}\frac{\partial}{\partial \overline{z}_2}(f_1''+f_2''\mathbf{j})\\
&=\Big((f_1'+f_2'\mathbf{j})\Big(\frac{\partial}{\partial \overline{z}_1}\Big)+(\overline{f}_1'+\overline{f}_2'\mathbf{j})\mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)(f_1''+f_2''\mathbf{j})\\
&=\Big(f'\Big(\frac{\partial}{\partial \overline{z}_1}\Big)+\overline{f}'\mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)f''.
\end{aligned}$$

■

3. Almost everywhere hyperholomorphic functions whose inverses are almost everywhere hyperholomorphic.

3.1. Definitions. We call *inverse* of a quaternionic function $f:q\mapsto f(q)$, the function defined almost everywhere on U: $q\mapsto f(q)^{-1}$; then: $f^{-1}=|f|^{-1}\overline{f}$, where $\overline{f}$ is the (quaternionic) conjugate of f, then: $f^{-1}=|f|^{-1}(\overline{f}_1-f_2\mathbf{j})$.

Behavior of f^{-1} at $q\in Z(f)$. Let $n_1=supJ_q^\alpha(|f|\overline{f}_1^{-1})$; $n_2=supJ_q^\alpha(|f|\overline{f}_2^{-1})$.

Define : $n_q=sup\ n_i$, $i=1,2$ as the *order of the pole* q *of* f^{-1}.

3.2. Characterisation.

PROPOSITION 3.1. *The following conditions are equivalent:*

(i) the function f and its right inverse are hyperholomorphic, when they are defined;

(ii) we have the equations:

$$(\overline{f}_1-f_1)\frac{\partial \overline{f}_1}{\partial z_1}-\overline{f}_2\frac{\partial f_2}{\partial z_1}-f_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}=0$$

$$\overline{f}_2\frac{\partial f_1}{\partial z_1}+\frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}=0$$

Proof. Let $f=f_1+f_2\mathbf{j}$ be a hyperholomorphic function and $g=g_1+g_2\mathbf{j}=|f|^{-1}(\overline{f}_1-f_2\mathbf{j})$

its inverse; so $g_1 = |f|^{-1}\overline{f}_1$; $g_2 = -|f|^{-1}f_2$, where $|f| = (f_1\overline{f}_1 + f_2\overline{f}_2)$.

$$\mathcal{D}g(q) = \frac{1}{2}\Big(\frac{\partial}{\partial \bar z_1} + \mathbf{j}\frac{\partial}{\partial \bar z_2}\Big)g(q) = \frac{1}{2}\Big(\frac{\partial g_1}{\partial \bar z_1} - \frac{\partial \bar g_2}{\partial z_2}\Big)(q) + \mathbf{j}\frac{1}{2}\Big(\frac{\partial g_1}{\partial \bar z_2} + \frac{\partial \bar g_2}{\partial z_1}\Big)(q)$$

$$\frac{\partial g_1}{\partial \bar z_1} = |f|^{-1}\frac{\partial \overline{f}_1}{\partial \bar z_1} - |f|^{-2}\overline{f}_1\Big(\frac{\partial f_1}{\partial \bar z_1}\overline{f}_1 + f_1\frac{\partial \overline{f}_1}{\partial \bar z_1} + \frac{\partial f_2}{\partial \bar z_1}\overline{f}_2 + f_2\frac{\partial \overline{f}_2}{\partial \bar z_1}\Big)$$

$$-\frac{\partial \bar g_2}{\partial z_2} = |f|^{-1}\frac{\partial \overline{f}_2}{\partial z_2} - |f|^{-2}\overline{f}_2\Big(\frac{\partial f_1}{\partial z_2}\overline{f}_1 + f_1\frac{\partial \overline{f}_1}{\partial z_2} + \frac{\partial f_2}{\partial z_2}\overline{f}_2 + f_2\frac{\partial \overline{f}_2}{\partial z_2}\Big)$$

$$\frac{\partial g_1}{\partial \bar z_2} = |f|^{-1}\frac{\partial \overline{f}_1}{\partial \bar z_2} - |f|^{-2}\overline{f}_1\Big(\frac{\partial f_1}{\partial \bar z_2}\overline{f}_1 + f_1\frac{\partial \overline{f}_1}{\partial \bar z_2} + \frac{\partial f_2}{\partial \bar z_2}\overline{f}_2 + f_2\frac{\partial \overline{f}_2}{\partial \bar z_2}\Big)$$

$$\frac{\partial \bar g_2}{\partial z_1} = -|f|^{-1}\frac{\partial \overline{f}_2}{\partial z_1} + |f|^{-2}\overline{f}_2\Big(\frac{\partial f_1}{\partial z_1}\overline{f}_1 + f_1\frac{\partial \overline{f}_1}{\partial z_1} + \frac{\partial f_2}{\partial z_1}\overline{f}_2 + f_2\frac{\partial \overline{f}_2}{\partial z_1}\Big)$$

$$\begin{aligned}2|f|^2\mathcal{D}g &\\ = (f_1\overline{f}_1 + f_2\overline{f}_2)\Big(\frac{\partial \overline{f}_1}{\partial \bar z_1} + \frac{\partial \overline{f}_2}{\partial z_2}\Big) &- \overline{f}_1 f_1\frac{\partial \overline{f}_1}{\partial \bar z_1} - \overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial \bar z_1} - \overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \bar z_1} - \overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial \bar z_1}\\ -\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_2} - f_1\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_2} &- \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_2} - f_2\overline{f}_2\frac{\partial \overline{f}_2}{\partial z_2}\\ +\mathbf{j}\Big((f_1\overline{f}_1 + f_2\overline{f}_2)\Big(\frac{\partial \overline{f}_1}{\partial \bar z_2} - \frac{\partial \overline{f}_2}{\partial z_1}\Big) &- \overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial \bar z_2} - \overline{f}_1 f_1\frac{\partial \overline{f}_1}{\partial \bar z_2} - \overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial \bar z_2} - \overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \bar z_2}\\ +\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1} + f_1\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1} &+ \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1} + f_2\overline{f}_2\frac{\partial \overline{f}_2}{\partial z_1}\Big)\end{aligned}$$

Use the fact: f is hyperholomorphic:

$$\frac{\partial f_1}{\partial \bar z_1} - \frac{\partial \overline{f}_2}{\partial z_2} = 0;\ \ \frac{\partial f_1}{\partial \bar z_2} + \frac{\partial \overline{f}_2}{\partial z_1} = 0 \tag{1}$$

$$\begin{aligned}&2|f|^2\mathcal{D}g =\\ &f_1\overline{f}_1\frac{\partial \overline{f}_2}{\partial z_2} + f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \bar z_1} - \overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial \bar z_1} - \overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \bar z_1} - \overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_2} - \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_2} + \overline{f}_2\frac{\partial f_2}{\partial \bar z_1}(f_1 - \overline{f}_1) +\\ &+\mathbf{j}\Big(+f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \bar z_2} - \overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial \bar z_2} - \overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \bar z_2} + \overline{f}_1\frac{\partial f_1}{\partial \bar z_2}(f_1 - \overline{f}_1) + \overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1} + f_1\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1} + \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}\Big)\end{aligned}$$

f being hyperholomorphic, g hyperholomorphic is equivalent to the system of two equations:

$$+f_1\overline{f}_1\frac{\partial \overline{f}_2}{\partial z_2} + f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \bar z_1} - \overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial \bar z_1} - \overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \bar z_1} - \overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_2} - \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_2} + \overline{f}_2\frac{\partial f_2}{\partial \bar z_1}(f_1 - \overline{f}_1) = 0$$

$$+f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \bar z_2} - \overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial \bar z_2} - \overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \bar z_2} + \overline{f}_1\frac{\partial f_1}{\partial \bar z_2}(f_1 - \overline{f}_1) + \overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1} + f_1\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1} + \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1} = 0$$

f_1 and f_2 satisfy, by conjugation of the second equation:

$$+f_2\overline{f}_2\frac{\partial f_1}{\partial z_1} - f_1 f_1\frac{\partial \overline{f}_1}{\partial z_1} - f_1\overline{f}_2\frac{\partial f_2}{\partial z_1} + f_2\frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1 - f_1) + f_1\overline{f}_1\frac{\partial f_2}{\partial \bar z_2} - f_1 f_2\frac{\partial \overline{f}_1}{\partial \bar z_2} - f_2 f_2\frac{\partial \overline{f}_2}{\partial \bar z_2} = 0$$

$$+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1}.+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}-\overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial \overline{z}_2}-\overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}+\overline{f}_1\frac{\partial f_1}{\partial \overline{z}_2}(f_1-\overline{f}_1)=0$$

Using (1), we get:

$$+f_2\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1(\overline{f}_1-f_1)\frac{\partial \overline{f}_1}{\partial z_1}-f_1\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_1 f_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}-f_2 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}=0$$

$$+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+(f_1-\overline{f}_1)\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}+\overline{f}_1\frac{\partial f_1}{\partial \overline{z}_2}(f_1-\overline{f}_1)+f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}.-\overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}=0$$

Assume $f_1\neq 0,\ f_2\neq 0$

$$\overline{f}_1\big(f_2\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1(\overline{f}_1-f_1)\frac{\partial \overline{f}_1}{\partial z_1}-f_1\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_1 f_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}-f_2 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}\big)=0$$

$$-f_2\big(+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+(f_1-\overline{f}_1)\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}-\overline{f}_1\frac{\partial \overline{f}_2}{\partial z_1}(f_1-\overline{f}_1)+f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}-\overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}\big)=0$$

By sum:

$$\overline{f}_1\big(f_1(\overline{f}_1-f_1)\frac{\partial \overline{f}_1}{\partial z_1}-f_1\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_1 f_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}\big)$$

$$-f_2\big((f_1-\overline{f}_1)\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}-\overline{f}_1\frac{\partial \overline{f}_2}{\partial z_1}(f_1-\overline{f}_1)+f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}\big)=0$$

i.e.

$$(\overline{f}_1 f_1+f_2\overline{f}_2)\big((\overline{f}_1-f_1)\frac{\partial \overline{f}_1}{\partial z_1}-\overline{f}_2\frac{\partial f_2}{\partial z_1}-f_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}\big)=0$$

$$\overline{f}_2\big(f_2\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1(\overline{f}_1-f_1)\frac{\partial \overline{f}_1}{\partial z_1}-f_1\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_1 f_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}-f_2 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}\big)=0$$

$$f_1\big(+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+(f_1-\overline{f}_1)\overline{f}_2\frac{\partial \overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}-\overline{f}_1\frac{\partial \overline{f}_2}{\partial z_1}(f_1-\overline{f}_1)+f_2\overline{f}_2\frac{\partial \overline{f}_1}{\partial \overline{z}_2}-\overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}\big)=0$$

By sum

$$\overline{f}_2\big(f_2\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_2\frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_2 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}\big)$$

$$+f_1\big(\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}-\overline{f}_1\frac{\partial \overline{f}_2}{\partial z_1}(f_1-\overline{f}_1)-\overline{f}_1 f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}\big)=0$$

i.e.

$$\overline{f}_2\frac{\partial f_1}{\partial z_1}+\frac{\partial \overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_2\frac{\partial \overline{f}_2}{\partial \overline{z}_2}=0$$

■

3.3. Definition. We will call *w-hypermeromorphic function* (w- for *weak*) any almost everywhere defined hyperholomorphic function whose right inverse is hyperholomorphic almost everywhere.

4. On the spaces of hypermeromorphic functions.

4.1. Sum of two w-hypermeromorphic functions.

PROPOSITION 4.1. *If f and g are two w-hypermeromorphic functions, then the following conditions are equivalent:*

(i) the sum $h = f + g$ is w-hypermeromorphic;
(ii) h satisfies the following PDE:

$$-(\frac{\partial|h|}{\partial\bar{z}_1} + \mathbf{j}\frac{\partial|h|}{\partial\bar{z}_2})(\bar{h}_1 - h_2\mathbf{j}) + |h|(\frac{\partial}{\partial\bar{z}_1} + \mathbf{j}\frac{\partial}{\partial\bar{z}_2})(\bar{h}_1 - h_2\mathbf{j}) = 0$$

Proof. Explicit the condition:

$$|h|^2\mathcal{D}(h^{-1}) = -\mathcal{D}(|h)|)(\bar{h}) + |h|\mathcal{D}(\bar{h}) = 0;$$

with $\bar{h} = \bar{h}_1 - h_2\mathbf{j}$

$$2\mathcal{D}\bar{h} = (\frac{\partial}{\partial\bar{z}_1} + \mathbf{j}\frac{\partial}{\partial\bar{z}_2})(\bar{h}_1 - h_2\mathbf{j}) = \frac{\partial\bar{h}_1}{\partial\bar{z}_1} + \frac{\partial\bar{h}_2}{\partial z_2} - (\frac{\partial h_2}{\partial\bar{z}_1} - \frac{\partial h_1}{\partial z_2})\mathbf{j}$$

$$\begin{aligned}\mathcal{D}(|h|) = \mathcal{D}(h_1\bar{h}_1 + h_2\bar{h}_2) &= \frac{1}{2}(\frac{\partial}{\partial\bar{z}_1} + \mathbf{j}\frac{\partial}{\partial\bar{z}_2})(h_1\bar{h}_1 + h_2\bar{h}_2)\\ &= \frac{1}{2}(\bar{h}_1\frac{\partial h_1}{\partial\bar{z}_1} + \bar{h}_2\frac{\partial h_2}{\partial\bar{z}_1} + h_1\frac{\partial\bar{h}_1}{\partial\bar{z}_1} + h_2\frac{\partial\bar{h}_2}{\partial\bar{z}_1})\\ &\quad + \frac{1}{2}(\bar{h}_1\frac{\partial h_1}{\partial z_2} + \bar{h}_2\frac{\partial h_2}{\partial z_2} + h_1\frac{\partial\bar{h}_1}{\partial z_2} + h_2\frac{\partial\bar{h}_2}{\partial z_2})\mathbf{j} = 0.\end{aligned}$$

■

4.2. Product of two w-hypermeromorphic functions.

PROPOSITION 4.2. *Let f, g be two w-hypermeromorphic functions on U, then the following conditions are equivalent:*

(i) the product $f * g$ is w-hypermeremorphic;
(ii) f and g satisfy the system of PDE:

$$g_1(\frac{\partial f_1}{\partial\bar{z}_1} + \frac{\partial\bar{f}_2}{\partial z_2}) + (f_1 - \bar{f}_1)\frac{\partial g_1}{\partial\bar{z}_1} + \bar{f}_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial\bar{g}_2}{\partial\bar{z}_1} = 0$$

$$g_1(\frac{\partial f_1}{\partial\bar{z}_2} - \frac{\partial\bar{f}_2}{\partial z_1}) + (f_1 - \bar{f}_1)\frac{\partial g_1}{\partial\bar{z}_2} - \bar{f}_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial\bar{g}_2}{\partial\bar{z}_2} = 0$$

Proof. Let $f = f_1 + f_2\mathbf{j}$ and $g = g_1 + g_2\mathbf{j}$ two hypermeromorphic functions and $f * g =$

$f_1g_1 - f_2\overline{g}_2 + (f_1g_2 - f_2\overline{g}_1)\mathbf{j}$ their product, then

$$\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2} = 0;$$

$$\frac{\partial(f_1g_1 - f_2\overline{g}_2)}{\partial \overline{z}_1} - \frac{\partial(\overline{f}_1\overline{g}_2 - \overline{f}_2g_1)}{\partial z_2}$$

$$= g_1(\frac{\partial f_1}{\partial \overline{z}_1} + \frac{\partial \overline{f}_2}{\partial z_2}) - \overline{g}_2(\frac{\partial \overline{f}_1}{\partial z_2} + \frac{\partial f_2}{\partial \overline{z}_1}) + f_1\frac{\partial g_1}{\partial \overline{z}_1} - \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_2} + \overline{f}_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_1} = 0$$

$$g_1(\frac{\partial f_1}{\partial \overline{z}_1} + \frac{\partial \overline{f}_2}{\partial z_2}) + f_1\frac{\partial g_1}{\partial \overline{z}_1} - \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_2} + \overline{f}_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_1} = 0.$$

$$\frac{\partial(f_1g_1 - f_2\overline{g}_2)}{\partial \overline{z}_2} + \frac{\partial(\overline{f}_1\overline{g}_2 - \overline{f}_2g_1)}{\partial z_1}$$

$$= g_1(\frac{\partial f_1}{\partial \overline{z}_2} - \frac{\partial \overline{f}_2}{\partial z_1}) + \overline{g}_2(\frac{\partial \overline{f}_1}{\partial z_1} - \frac{\partial f_2}{\partial \overline{z}_2}) + f_1\frac{\partial g_1}{\partial \overline{z}_2} - \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_1} + \overline{f}_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_2} = 0$$

$$g_1(\frac{\partial f_1}{\partial \overline{z}_2} - \frac{\partial \overline{f}_2}{\partial z_1}) + f_1\frac{\partial g_1}{\partial \overline{z}_2} + \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_1} - \overline{f}_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_2} = 0$$

■

4.3. Definition. We will call *hypermeromorphic* the w-hypermeromorphic functions whose sum and product are w-hypermeromorphic. Their space is nonempty, since it contains the space of the meromorphic functions.

5. Globalisation. Hamilton 4-manifold.

5.1. The hypermeromorphic functions on a relatively compact open set U of $\mathbb{H}$ play the part of the meromorphic functions on a relatively compact open set U of $\mathbb{C}$. We will call *pseudoholomorphic function on* U, every hypermeromorphic function, without poles on U. We will call *smooth hypermeromorphic function (sha function) on* U, every hypermeromorphic function, without zeroes and poles on U.

LEMMA 5.1. *The quotient of two pseudoholomorphic functions on* U*, with the same zeroes and the same orders, is a sha function on* U*.*

5.2. Manifolds. The sha functions have been defined on open sets of $\mathbb{H} \cong \mathbb{C}^2$. Let X be a 4-dimensional manifold bearing an atlas $\mathcal{A}$ of charts (h_j, U_j) such as the transition functions $h_{i,j} : U_i \cap U_j \to \mathbb{H}$ are sha functions. $X = (X, \mathcal{A})$ will be called an $\mathcal{A}$-manifold analogous for $\mathbb{H}$ of a Riemann surface for $\mathbb{C}$. I also propose to call an $\mathcal{A}$-manifold a *Hamilton 4-manifold.*

5.3. Sheaves of pseudoholomorphic, hypermeromorphic functions.

5.3.1. *Functions on an* $\mathcal{A}$*-manifold* $X = (X, \mathcal{A})$. A map $f : X \to \mathbb{H}$ is called a *pseudo-holomorphic function* on X, if it is continuous and satisfies the following condition: for every chart (h, U) of X, $(f|U)h^{-1} : h(U) \to \mathbb{H})$ is a pseudoholomorphic. In the same way, a map $f : X \to \mathbb{H}$ is called a *hypermeromorphic function* on X, if it is continuous and satisfies the following condition:for every chart (h, U) of X, $(f|U)h^{-1} : h(U) \to \mathbb{H})$ is a hypermeromorphic.

5.3.2. *Examples of Hamilton 4-manifold.* The identity map of $\mathbb{H}$ is: $z_1 + z_2\mathbf{j} \mapsto z_1 + z_2\mathbf{j}$.
Ex. 1: $(id_{\mathbb{H}}, \mathbb{H})$ is the unique chart of the atlas defining $\mathbb{H}$ as an $\mathcal{A}$-manifold. *Proof.* The identity map ($id_{\mathbb{H}}$ is $f_1 = z_1, f_2 = z_2$ is pseudoholomorphic.
Ex.2: Every open set V of X bears an induced structure of *Hamilton 4-manifold.*
Ex. 3: *Hamilton hypersphere* $\mathbb{HP}$.

In the space $\mathbb{H}\mathrm{x}\mathbb{H} \setminus \{0\}$, consider the equivalence relation $\rho_1 \mathcal{R} \rho_2$: there exists $\lambda \in \mathbb{H}^* = \mathbb{H} \setminus 0$ such that $\rho_2 = \rho_1 \lambda$ (right multiplication by λ). The elements of $\mathbb{H}\mathrm{x}\mathbb{H} \setminus \{0\}$ are the pairs $(q_1, q_2) \neq (0, 0)$. Let

$\pi : \mathbb{H}\mathrm{x}\mathbb{H} \setminus \{0\} \to \big(\mathbb{H}\mathrm{x}\mathbb{H} \setminus \{0\}\big)/\mathcal{R}$ denoted $\mathbb{HP}$.

$(q_1, q_2) \mapsto$ class of (q_1, q_2)

So, $\mathbb{HP}$ is the set of the quaternionic lines from the origin of $\mathbb{H}^2$.

Consider the pairs $(q_1, q_2) \in \mathbb{H}^2$, with $q_2 \neq 0$ we have: $\pi(q_1, q_2) = \pi(q_1 q_2^{-1}, 1)$; let $\zeta = q_1 q_2^{-1}, q_2 \neq 0$; in the same way, consider the pairs $(q_1, q_2) \in \mathbb{H}^2$, with $q_1 \neq 0$ we have: $\pi(q_1, q_2) = \pi(1, q_2 q_1^{-1})$; let $\zeta' = q_2 q_1^{-1}, q_1 \neq 0$. The charts ζ, ζ' have for domains U, U', two open sets of $\mathbb{HP}$, respectively homeomorphic to $\mathbb{H}$ forming an atlas of $\mathbb{HP}$. Remark that U covers the whole of $\mathbb{HP}$ except the point $\pi(q_1, 0)$ denoted ∞, and that U' covers the whole of $\mathbb{HP}$ except the point $\pi(0, q_2)$ denoted 0. $U' = \mathbb{HP} \setminus \{0\}$. Over $U \cap U'$, we have: $\zeta.\zeta' = 1$, i.e. $\zeta' = \zeta^{-1}$ and $\zeta = q_1 q_2^{-1}$.

5.3.3. *Pseudoholomorphic map* or *morphism.*

Let X and Y be two Hamilton 4-manifolds, a map $f : X \to Y$ is said pseudoholomorphic if it is continuous and if, for every pair of pseudoholomorphic charts $(h, U), (k, V)$ such that $f(U) \subset V$,

$k(f|U)h^{-1} : h(U) \to k(V)$ be pseudoholomorphic.

5.3.4. *Sheaf of pseudoholomorphic functions.* Let U, V be two open sets of X such that $U \subset V$, then, the restrictions to U of the pseudoholomorphic functions on V are pseudoholomorphic on U.

So is defined the *sheaf,* denoted $\mathcal{P}$, of (non commutative rings) *of pseudoholomorphic functions on* X. The pair (X,$\mathcal{P}$) is a *ringed space.*

In the same way, the *sheaf* of non commutative rings, denoted $\mathcal{M}$, *of hypermeromorphic functions is defined on* X.

5.3.5. *Hamiltonian Submanifolds.* They are submanifolds whose function ring is pseudoholomorphic. We will implicitly use the following fact: If f is a pseudoholomorphic or hypermeromorphic function, the same is true for $a + f$, where a is any fixed quaternion.

The following examples are complex analytic submanifolds.

i) $\mathbb{H}$. Let a be a fixed quaternion, then $a + \mathbb{C} \subset \mathbb{H}$ *is a complex line from a embedded in* $\mathbb{H}$.

ii) $\mathbb{HP}$. *Complex projective line imbedded in* $\mathbb{HP}$. Let $i : z_1 \mapsto z_1 + z_2\mathbf{j}$ and $j : \mathbb{CP} \mapsto \mathbb{HP}$

$$\begin{array}{lcl} \mathbb{C} \times \mathbb{C} \setminus 0 & \to & \mathbb{H} \times \mathbb{H} \setminus \{0\} \\ i \times i \downarrow & & \downarrow \\ \big(\mathbb{C} \times \mathbb{C} \setminus 0\big)/\mathcal{R}' & \to & \big(\mathbb{H} \times \mathbb{H} \setminus \{0\}\big)\mathcal{R} \end{array}$$

Let $p \in \mathbb{HP}$ be a fixed point. Then, $p + \mathbb{C}P^1$ is a *complex projective line* (or Riemann sphere) *from p, embedded in* $\mathbb{HP}$.

iii) Let S be a compact Riemann surface contained in $\mathbb{HP}$ as a Hamiltonian submanifold. Then $p+S$ is a *compact Riemann surface from* p*, embedded in* $\mathbb{HP}$.

5.3.6. *A family of complex submanifolds in a Hamilton 4-manifold.* We now use the properties and notions of subsection 2.5.2. They are $a+A_{\alpha,\beta}$ and also for restrtictions to an open set U of $\mathbb{H}$.

On a Hamilton 4-manifold X with an atlas $\mathcal{A}$ and every domain of chart U as above, we obtain:

Proposition 5.2. *Let $(X, \mathcal{P})$ be a Hamilton 4-manifold. There exist a family of complex analytic curves $C_{b,\gamma,\delta}$, of X. For every U domain of coordinates in $\mathcal{A}$ let $A_{\gamma,\delta}$. By gluing, we get a complex analytic curve in $(X, \mathcal{P})$ from $b \in X$, and γ, δ are real parameters.*

Proof. Let $b \in X; \beta, \gamma \in \mathbb{R}$ be given, consider an atlas $\mathcal{A}$ whose domains of charts are either open sets U of X disjoint from $A_{\beta,\gamma}$, or $V_{\beta,\gamma} = U \cup (A_{\beta,\gamma} \cap U)$ where $A_{\beta,\gamma} \cap U$ is connected, not empty. The restrictions of the charts of $\mathcal{A}$ to the $U \cup (A_{\beta,\gamma} \cap U)$ define an atlas of $C_{b,\gamma,\delta}$ as complex analytic subvariety of $(X, \mathcal{P})$, in the following way: assume $b \in V_{\beta,\gamma} \cap A_{\beta,\gamma} \cap U$ and consider the open sets analogous to $V_{\beta,\gamma}$ such that the various $V_{\beta,\gamma}$ be connected. Then the corresponding $A_{\beta,\gamma} \cap U$ constitute a covering of the unique complex analytic curve $C_{b,\gamma,\delta}$. ∎

5.3.7. Let C be a complex analytic curve embedded into X and an atlas $\mathcal{A}$ such that every chart of domain U meeting C satisfies: $U \cap C$ is connected

Theorem 5.3. *The set of complex analytic curves in X is the family $C_{b,\gamma,\delta}$.*

6. Hamilton 4-manifold of a hypermeromorphic function.

6.1. Analytic continuation along a path. [D90, p. 116]

Let X be a Hamilton 4-manifold, $\gamma : [0,1] \to X$ a continuous path from a to b, $\varphi \in \mathcal{P}_a$ a germ of pseudoholomorphic function at a.

Let $\tau \in [0,1]$ and $\varphi_\tau \in \mathcal{P}_{\gamma(\tau)}$, there exists an open neighborhood U_τ of $\gamma(\tau)$ in X and a pseudoholomorphic function $f_\tau \in \mathcal{P}(U_\tau)$ such that $\rho^{U_\tau}_{\gamma(\tau)} f_\tau = \varphi_\tau$. γ being continuous, it exists an open neighborhood W_τ of τ in [0,1] such that $\gamma(W_\tau) \subset U_\tau$.

6.2. Definition. A germ $\psi \in \mathcal{P}_b$ is said to be *the analytic continuation of* φ *along* γ if there exists a family $(\varphi_t)_{t\in[0,1]}$ such that:

1) $\varphi_0 = \varphi$ and $\varphi_1 = \psi$.
2) for every $\tau \in [0,1]$, for every $t \in W_\tau$, we have: $\rho^{U_\tau}_{\gamma(\tau)} f_\tau = \varphi_\tau$

Theorem 6.1. *Identity theorem. Let X be a connected Hamilton 4-manifold and $f_1, f_2 : X \to Y$ be two morphisms which coincide in the neighborhood of a point $x_0 \in X$, then f_1, f_2 coincide on X.*

Proof as for Riemann surfaces, [D90, ch. 5].

Theorem 6.2. *Let X be a simply connected Hamilton 4-manifold, $a \in X$, $\varphi \in \mathcal{P}_a$ be a germ having an analytic continuation along every path from a. Then there exists a unique function $f \in \mathcal{P}(X)$ such that $\rho^X_a f = \varphi$.*

(cf. [D90, ch. 5, 4.1.5])

Let $p: Y \to X$ be a morphism of two Hamilton 4-manifolds; p is locally bi-pseudoholomorphic, then it defines, for every $y \in Y$, an isomorphism $p_y : \mathcal{P}_{x,p(y)} \to \mathcal{P}_{Y,y}$; this defines: $p_* = p_{*y} = (p_y^*)^{-1}$.

6.3. Definition. Let X be a Hamilton 4-manifold, $a \in X$, $\varphi \in \mathcal{P}_a$. A quadruple (Y, p, f, b) is called an *analytic continuation of* φ if:

(i) Y is a Hamilton 4-manifold, $p: Y \to X$ is a morphism;

(ii) f is a pseudoholomorphic function on Y;

(iii) $b \in p^{-1}(a) \subset Y$; $p_*(\rho_b^Y f) = \varphi$.

An analytic continuation is said to be *maximal* if it is solution of the following universal map problem: for every analytic continuation (Z, q, g, c) of φ, there exists a fibered morphism $F: Z \to Y$ such that $F(c) = b$ and $F^*(f) = g$. Hence

If (Y, p, f, b) is a maximal analytic continuation of φ, it is unique up to an isomorphism. Y is called the Hamilton 4-manifold of φ.

Theorem 6.3. *Let X be a Hamilton 4-manifold, $a \in X$, $\varphi \in \mathcal{P}_a$. Then there exists a maximal analytic continuation of φ.*

6.4. Remark. Then, we will say that the above function f is *the unique maximal analytic continuation of the germ φ*. Moreover, the above definitions and results of the section 2 are valid for the sheaf $\mathcal{M}$ of hypermeromorphic functions instead of the sheaf $\mathcal{P}$.

6.5. Main result.

Theorem 6.4. *Let X be a Hamilton 4-manifold and $P(T) = T^n + c_1 T^{n-1} + \ldots + c_n \in \mathcal{M}(X)[T]$ be an irreducible polynomial of degree n. Then there exist a Hamilton 4-manifold Y, a ramified pseudoholomorphic covering (cf. [D90, ch. 5] for Riemann surfaces) with n leaves $\Pi: Y \to X$ and a hypermeromorphic function $F \in \mathcal{M}(Y)$ such that $(\Pi^* P)(F) = 0$.*

F is the unique maximal analytic continuation of every hypermeromorphic germ φ of X such that $P(\varphi) = 0$; F is called the *hyperalgebraic function defined by the polynomial* P and Y is the *Hamilton 4-manifold of F*.

Proof at the end of the section.

1) *X is compact connected.*

2) *Every pseudoholomorphic function on X is constant.*

3) *Every hypermeromorphic function f on X different from ∞ is rational.*

4) *In case $X = \mathbb{HP}$, in Theorem 6.4, c_j is rational.* Indeed, since c_j is hypermeromorphic, from 3), it is rational.

6.6. Proof of Theorem 6.4. In the notations of Ex. 3, ζ is a local coordinate on $X = \mathbb{HP}$.

f has a finite set of poles $p_1, \ldots, p_n$. Assume that ∞ is not a pole of f, then $p_1, \ldots, p_n \in \mathbb{H}$. Let h_ν the principal part of f at p_ν, then $f - h_\nu = a_\nu$, constant, from 2) and $h_\nu = \sum_{j=-k_\nu}^{-1} C_{\nu j}(\zeta - p_\nu^j)$ is a hypermeromorphic function, where $C_{\nu j} \in \mathbb{H}$.

6.6.1. *Elementary symmetric functions.* Let

$$\Pi : Y \to X$$

be a nonramified pseudoholomorphic covering with n leaves, and f be a hypermeromorphic function on Y. Every point $x \in X$ has an open neighborhood U such that $\Pi^{-1}(U) = \bigcup_{j=1}^{n} V_j$ where the V_j are disjoint and $\Pi|V_j : V_j \to U$ is bi-pseudoholomorphic, $(j = 1, \ldots, n)$; let $\varphi_j : U \to V_j$ the reverse (i.e. set inverse) of $\Pi|V_j$ and $f_j = \varphi_j^* f = f.\varphi_j$. Then:

$$\Pi_{j=1}^{n}(T - f_j) = T^n + c_1 T^{n-1} + \ldots + c_n;$$

$c_j = (-1)^j s_j(f_1, \ldots, f_n)$, where s_j is the j-th elementary symmetric function in n variables. The c_j are hypermeromorphic, locally defined, but glue together into $c_1, \ldots, c_n \in \mathcal{M}(X)$ and are called *the elementary symmetric functions of* f *with respect to* Π.

6.6.2. *Remark.* The elementary symmetric functions of a hypermeromorphic function on Y are still defined when the covering Π is ramified.

6.6.3.

THEOREM 6.5. *Let* Π *as in Theorem 6.4, with* Y *not necessarily connected,* $A \subset X$ *be a discreet closed subset containing all the critical values of* Π*, and* $B = \Pi^{-1}(A)$*.*

Let f *be a pseudoholomorphic (resp. hypermeromorphic) function on* $Y \setminus B$ *and*

$$c_1, \ldots, c_n \in \mathcal{H}(X \setminus A)(resp. \mathcal{M}(X \setminus A))$$

the elementary symmetric functions of f*. Then the following two conditions are equivalent:*

(i) f *has a pseudoholomorphic (resp. hypermeromorphic) extension to* Y*;*

(ii) for every $j = 1, \ldots, n$*,* c_j *has a pseudoholomorphic (resp. hypermeromorphic) extension to* X*.*

6.6.4. *Existence of* Y *in Theorem 6.4.* Let $\Delta \in \mathcal{M}(X)$ be the discriminant of $P(T)$; $P(T)$ being irreducible, $\Delta \neq 0$: then there exists a discrete closed set $A \subset X$ such that, for every $x \in X' = X \setminus A$, $\Delta(x) \neq 0$, and all the functions c_j are pseudoholomorphic.

Let $Y' = \{\varphi \in \mathcal{H}_x, x \in X'; P(\varphi) = 0\} \subset L\mathcal{H}$, etal space defined by the sheaf $\mathcal{H}$, and $\Pi' : Y' \to X$, $(\varphi \mapsto x)$.

It can be shown that, for every $x \in X'$, there exists an open neighborhood U of x in X' and functions $f_j \in \mathcal{H}(U)$, $j = 1, \ldots, n$, such that $P(T)|U = \Pi_{j=1}^{n}(T - f_j)$; then $\Pi'^{-1}(U) = \bigcup_{j=1}^{n}[U, f_j]$ where $[U, f_j] = \{f_{jy}, y \in U\}$ is an open set of $L\mathcal{H}$ and $\Pi'|[U, f_j] : [U, f_j] \to U$ is a homeomorphism; Y' is a Hamilton 2-manifold non necessarily connected, and a pseudoholomorphic, non ramified covering of X'. It can be shown that Π' can be extended into a ramified pseudoholomorphic covering $\Pi : Y \to X$ of X for which $Y' = \Pi^{-1}(X')$.

The c_j are defined on the whole of X; from Theorem 6.5, f has an extension $F \in \mathcal{M}(X)$ such that

$$\Pi^* P(F) = F^n + (\Pi^* c_1)F^{n-1} + \ldots + \Pi^* c_n = 0.$$

It is easy to prove the connectedness of Y and the unicity of F.

This ends the proof of Theorem 6.4.

7. The Hamilton 4-manifold Y of F when $X = \mathbb{HP}$.

7.1. Recall the main properties of Y.

Y is of real dimension 4;

Y is connected;

Y is compact;

Y is C^∞;

let m be the number of the critical values of Π and q_j these critical values; they define points of Y forming the 0-skeleton of a simplicial complex K carried by the manifold Y. K may be supposed to be C^∞ by parts. Cutting along the 3-faces of K defines a fundamental domain FD of the covering Π. FD is a 4-dim polytope in $\mathbb{HP}$ with an even number of 3-faces; gluying together the opposite 3-faces, we get a compact 4-dim polytope with homology of the Hamilton 4-manifold Y.

7.2. Homology of Y. $H^p(Y; \mathbb{Z})$, for $p = 0, \ldots, 4$ have to be evaluated, using the critical values q_j, and the Poincaré duality.

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